Leen Ammeraal · Kang Zhang

Computer Graphics for Java Programmers

Third Edition

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Preface

It has been 10 years since the publication of the second edition. The programming language, Java, has now developed into its maturity, being the language of choice in many industrial and business domains. Yet the skills of developing computer graphics applications using Java are surprisingly lacked in the computer science curricula. Though no longer active in classroom teaching, the first author has developed and published several Android applications using Java, the main language for Android developers. The second author has taught Computer Graphics at his current university for the past 17 years using the first and second editions of this textbook, apart from his previous years in Australia using different textbooks. We feel strongly a need for updating the book.

This third edition continues the main theme of the first two editions, that is, graphics programming in Java, with all the source code, except those for exercises, available to the reader. Major updates in this new edition include the following:

1. The contents of all chapters are updated according to the authors' years of classroom experiences and recent feedback from our students.
2. Hidden-line elimination and hidden-face elimination are merged into a single chapter.
3. A new chapter on color, texture, and lighting is added, as Chap. 7.
4. The companion software package, CGDemo, that demonstrates the working of different algorithms and concepts introduced in the book, is enhanced with two new algorithms added and a few bugs fixed.
5. A set of 37 video sessions (7–11 min each) in MOOC (Massive Open Online Course) style, covering all the topics of the textbook, is supplemented.
6. A major exercise, split into four parts, on implementing the game of Tetris is added at the end of four relevant chapters.

Many application examples illustrated in this book could be readily implemented using Java 3D or OpenGL without any understanding of the internal working of the implementation, which we consider undesirable for computer science students. We therefore believe that this textbook continues to serve as an indispensable introduction to the foundation of computer graphics, and more importantly, *how* various classic algorithms are designed. It is essential for computer science students to learn the skills on how to optimize time-critical algorithms and how to develop elegant algorithmic solutions.

The example programs can be downloaded from the Internet at:

http://home.kpn.nl/ammeraal/

or at:

http://www.utdallas.edu/~kzhang/BookCG/

Finally, we would like to thank the UT-Dallas colleague Pushpa Kumar, who has been using this textbook to teach undergraduate Computer Graphics class and provided valuable feedback. We are grateful to Susan Lagerstrom-Fife of Springer for her enthusiastic support and assistance in publishing this edition.

Kortenhoef, The Netherlands Leen Ammeraal
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Contents

Chapter 1
Elementary Concepts

This book is primarily about computer graphics programming and related mathematics. Rather than discussing general graphics subjects for end users or how to use graphics software, we will cover more fundamental subjects, required for graphics programming. In this chapter, we will first understand and appreciate the nature of discreteness of displayed graphics on computer screens. We will then see that x- and y-coordinates need not necessarily be pixel numbers, also known as device coordinates. In many applications, logical coordinates are more convenient, provided we can convert them to device coordinates before displaying on the screen. With input from a mouse, we would also need the inverse conversion, i.e. converting device coordinates to logical coordinates, as we will see at the end of this chapter.

1.1 Pixels and Device Coordinates

The most convenient way of specifying a line segment on a computer screen is by providing the coordinates of its two endpoints. In mathematics, coordinates are real numbers, but primitive line-drawing routines may require these to be integers. This is the case, for example, in the Java language, to be used throughout this book. The Java Abstract Windows Toolkit (AWT) provides the class *Graphics* containing the method *drawLine*, which we use as follows to draw the line segment connecting A and B:

```
g.drawLine(xA, yA, xB, yB);
```

The graphics context g in front of the method is normally supplied as a parameter of the *paint* method in the program, and the four arguments of *drawLine* are integers, ranging from zero to some maximum value. The above call to *drawLine* produces exactly the same line on the screen as this one:

```
g.drawLine(xB, yB, xA, yA);
```


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We will now use statements such as the above in a complete Java program. Fortunately, you need not type these programs yourself, since they are available from the Internet, as specified in the Preface. It will also be necessary to install the Java Development Kit (JDK). If you are not yet familiar with Java, you should consult other books, such as those mentioned in the Bibliography. This book assumes you to be fluent in basic Java programming.

The following program draws the largest possible rectangle in a canvas. The color red is used to distinguish this rectangle from the frame border:

```
// RedRect.java: The largest possible rectangle in red.
import java.awt.*;
import java.awt.event.*;

public class RedRect extends Frame {
   public static void main(String[] args) {new RedRect();}

   RedRect() {
      super("RedRect");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {Sytem.exit(0);}
      });
      setSize(300, 150);
      add("Center", new CvRedRect());
      setVisible(true);
   }
}

class CvRedRect extends Canvas {
   public void paint(Graphics g) {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      g.drawString("d.width  = " + d.width, 10, 30);
      g.drawString("d.height = " + d.height, 10, 60);
      g.setColor(Color.red);
      g.drawRect(0, 0, maxX, maxY);
   }
}
```

The call to *drawRect* almost at the end of this program has the same effect as these four lines:

```
g.drawLine(0, 0, maxX, 0);        // Top edge
g.drawLine(maxX, 0, maxX, maxY); // Right edge
g.drawLine(maxX, maxY, 0, maxY); // Bottom edge
g.drawLine(0, maxY, 0, 0);        // Left edge
```

The program contains two classes:

RedRect: The class for the frame, also used to close the application.
CvRedRect: The class for the canvas, in which we display graphics output.

However, after compiling the program by entering the command

```
javac RedRect.java
```

we notice that three class files have been generated: *RedRect.class*, *CvRedRect.class* and *RedRect*\$1*.class*. The third one is referred to as an *anonymous class* since it has no name in the program. It is produced by the following program segment:

```
addWindowListener(new WindowAdapter() {
   public void windowClosing(WindowEvent e) {System.exit(0);}
});
```

which enables the user of the program to terminate it in the normal way. The argument of the method *addWindowListener* must be an object of a class that implements the interface *WindowListener*. This implies that this class must define seven methods, one of which is *windowClosing*. The base class *WindowAdapter* defines these seven methods as do-nothing functions. In the above program segment, the argument of *addWindowListener* denotes an object of an anonymous subclass of *WindowAdapter*. In this subclass we override the method *windowClosing*.

The *RedRect* constructor shows that the frame size is set to 400×200. If we do not modify this size (by dragging a corner or an edge of the window), the canvas size is somewhat smaller than the frame. After compilation, we run the program by typing the command

```
java RedRect
```

which, with the given frame size, produces the largest possible red rectangle, shown in Fig. 1.1 just inside the frame.

The blank area in a frame, which we use for graphics output, is referred to as a *canvas*, which is a subclass, such as *CvRedRect* in program *RedRect.java*, of the AWT class *Canvas*. If, instead, we displayed the output directly in the frame, we

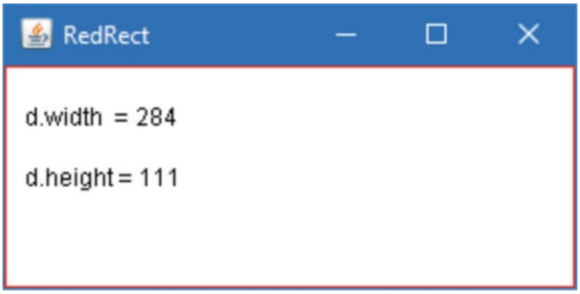


Fig. 1.1 Largest possible rectangle and canvas dimensions

would have a problem with the coordinate system: its origin would be in the top-left corner of the *frame*; in other words, the x-coordinates increase from left to right and y-coordinates from top to bottom. Although there is a method *getInsets* to obtain the widths of all four borders of a frame so that we could compute the dimensions of the client rectangle ourselves, we prefer to use a canvas.

The tiny screen elements that we can assign a color are called *pixels* (short for *picture elements*), and the integer x- and y-values used for them are referred to as *device coordinates*. Although there are 200 pixels on a horizontal line in the entire frame, only 192 of these lie on the canvas, the remaining 8 being used for the left and right borders. On a vertical line, there are 100 pixels for the whole frame, but only 73 for the canvas. Apparently, the remaining 27 pixels are used for the title bar and the top and bottom borders. Since these numbers may differ on different Java implementations and the user can change the window size, it is desirable that our program can determine the canvas dimensions. We do this by using the *getSize* method of the class *Component*, which is a superclass of *Canvas*. The following program lines in the paint method show how to obtain the canvas dimensions and how to interpret them:

```
Dimension d = getSize();
int maxX = d.width - 1, maxY = d.height - 1;
```

The *getSize* method of *Component* (a superclass of *Canvas*) supplies us with the numbers of pixels on horizontal and vertical lines of the canvas. Since we start counting at zero, the highest pixel numbers, *maxX* and *maxY*, on these lines are one less than these numbers of pixels. Remember that this is similar with arrays in Java and C. For example, if we write:

```
int[] a = new int[8];
```

the highest possible index value is 7, not 8. In such cases, the "index" is always one less than "size". Figure 1.2 illustrates this for a very small canvas, which is only 8 pixels wide and 4 high, showing a much enlarged screen grid structure. It also shows that the line connecting the points (0, 0) and (7, 3) is approximated by a set of eight pixels.

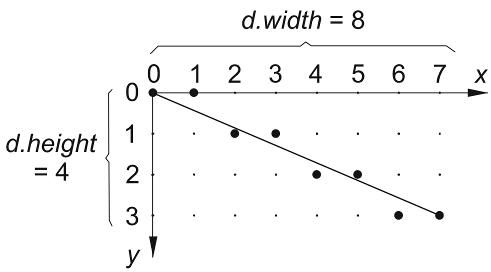


Fig. 1.2 Pixels as coordinates in a 8×4 canvas (with $maxX = 7$ and $maxY = 3$)

The big dots approximating the line denote pixels that are set to the foreground color. By default, this foreground color is black, while the background color is white. These eight pixels are made black as a result of this call:

```
g.drawLine(0, 0, 7, 3);
```

In the program *RedRect.java*, we used the following call to the *drawRect* method (instead of four calls to *drawLine*):

```
g.drawRect(0, 0, maxX, maxY);
```

In general, the call:

```
g.drawRect(x, y, w, h);
```

draws a rectangle with (x, y) as its top-left and $(x + w, y + h)$ as its bottom-right corners. In other words, the third and fourth arguments of the *drawRect* method specify the width and height, rather than the bottom-right corner, of the rectangle to be drawn. Note that this rectangle is $w + 1$ pixels wide and $h + 1$ pixels high. The smallest possible square, consisting of 2×2 pixels, is drawn by this call:

```
g.drawRect(x, y, 1, 1);
```

To put only one pixel on the screen, we cannot use *drawRect*, because nothing at all appears if we try to set the third and fourth arguments of this method to zero. Curiously enough, Java does not provide a special method for this purpose, so we have to use this method:

```
g.drawLine(x, y, x, y);
```

Note that the method:

```
g.drawLine(xA, y, xB, y);
```

draws a horizontal line consisting of $|xB - xA| + 1$ pixels.

In mathematics, lines are continuous without thickness, but are discrete and at least one pixel thick in computer graphics output. This difference in the interpretation of the notion of lines may not cause any problems if the pixels are very small in comparison with what we are drawing. However, we should be aware that there may be such problems in special cases, as Fig. 1.3a illustrates. Suppose that we have to draw a filled square ABCD of, say, 4×4 pixels, consisting of the bottom-right triangle ABC and the upper-left triangle ACD, which we want to paint in dark gray and light gray, respectively, without drawing any lines. Strangely enough, it is not clear how this can be done: if we make the diagonal AC light gray, triangle ABC contains fewer pixels than triangle ACD; if we make AC dark gray, it is the other way round.

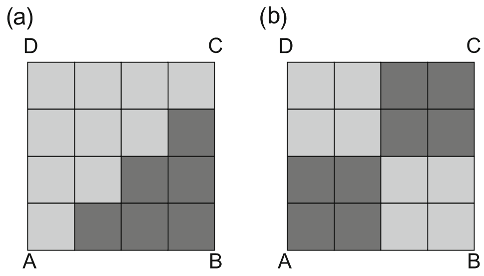


Fig. 1.3 Small filled regions

A much easier but still non-trivial problem, illustrated in Fig. 1.3b, is filling a checker-board with, say, dark and light gray squares instead of black and white ones. Unlike squares in mathematics, those on the computer screen deserve special attention with regard to the edges belonging or not belonging to the filled regions. We have seen that the call:

```
g.drawRect(x, y, w, h);
```

draws a rectangle with corners (x, y) and $(x + w, y + h)$. The method *fillRect*, on the other hand, fills a slightly smaller rectangle. The call:

```
g.fillRect(x, y, w, h);
```

assigns the current foreground color to a rectangle consisting of $w \times h$ pixels. This rectangle has (x, y) as its top-left and $(x + w - 1, y + h - 1)$ as its bottom-right corner. To obtain a generalization of Fig. 1.3b, the following method, *checker*, draws an $n \times n$ checker board, with (x, y) as its top-left corner and with dark gray and light gray squares, each consisting of $w \times w$ pixels. The bottom-left square will always be dark gray because for this square we have $i = 0$ and $j = n - 1$, so that $i + n - j = 1$:

```
void checker(Graphics g, int x, int y, int n, int w) {
   for (int i=0; i<n; i++)
      for (int j=0; j<n; j++) {
         g.setColor((i + n - j) % 2 == 0 ?
            Color.lightGray : Color.darkGray);
         g.fillRect(x + i * w, y + j * w, w, w);
      }
}
```

If we wanted to draw only the edges of each square, also in dark gray and light gray, we would have to replace the above call to *fillRect* with

```
g.drawRect(x + i * w, y + j * w, w - 1, w - 1);
```

in which the last two arguments are $w - 1$ instead of w.

1.2 Logical Coordinates

The Direction of the y*-axis*

As Fig. 1.2 shows, the origin of the device-coordinate systems lies at the top-left corner of the canvas, so that the positive y-axis points downward. This is reasonable for text output, that starts at the top and increases y as we go to the next line of text. However, this direction of the y-axis is different from typical mathematical practice and therefore often inconvenient in graphics applications. For example, in a discussion about a line with a positive slope, we expect to go upward when moving along this line from left to right. Fortunately, we can arrange for the positive y direction to be reversed by performing this simple transformation:

$$y' = maxY - y$$

Continuous Versus Discrete Coordinates

Instead of the discrete (integer) coordinates at the lower, device oriented level, we often wish to use continuous (floating-point) coordinates at the higher, problem-oriented level. Other usual terms are *device* and *logical* coordinates, respectively. Writing conversion routines to compute device coordinates from the corresponding logical ones and vice versa is a bit tricky. We must be aware that there are two solutions to this problem: *rounding* and *truncating*, even in the simple case in which increasing a logical coordinate by one results in increasing the device coordinate also by one. We wish to write the following methods:

$iX(x)$, $iY(y)$: converting the logical coordinates x and y to device coordinates;
$fx(x)$, $fy(y)$: converting the device coordinates X and Y to logical coordinates.

One may notice that we have used lower-case letters to represent logical coordinates and capital letters to represent device coordinates. This will be the convention used throughout this book. With regard to x-coordinates, the rounding solution could be:

```
int iX(float x){return Math.round(x);}
float fx(int x){return (float)x;}
```

For example, with this solution we have

$$iX(2.8) = 3 \quad \text{and} \quad fx(3) = 3.0$$

The truncating solution could be:

```
int iX(float x){return (int)x;}           // Not used in
float fx(int x){return (float)x + 0.5F;}  // this book.
```

With these conversion functions, we would have

$$iX(2.8) = 2 \text{ and } fx(3) = 2.5$$

We will use the rounding solution throughout this book, since it is the better choice if logical coordinates frequently happen to be integer values. In these cases the practice of truncating floating-point numbers will often lead to worse results than those with rounding.

Apart from the above methods *iX* and *fx* (based on rounding) for *x*-coordinates, we need similar methods for *y*-coordinates, taking into account the opposite directions of the two *y*-axes. At the bottom of the canvas the device *y*-coordinate is *maxY* while the logical *y*-coordinate is 0, which may explain the two expressions of the form *maxY* – . . . in the following methods:

```
int iX(float x){return Math.round(x);}
int iY(float y){return maxY - Math.round(y);}
float fx(int x){return (float)x;}
float fy(int y){return (float)(maxY - y);}
```

Figure 1.4 shows a fragment of a canvas, based on $maxY = 16$.

The pixels are drawn as black dots, each placed in the center of a square of dashed lines and the device-coordinates (X, Y) are placed between parentheses near each dot. For example, the pixel with device coordinates (8, 2), at the upper-right corner of this canvas fragment, has logical coordinates (8.0, 14.0). We have:

```
iX(8.0)   = Math.round(8.0) = 8
iY(14.0) = 16 - Math.round(14.0) = 2
fx(8)     = (float)8 = 8.0
fy(2)     = (float)(16 - 2) = 14.0
```

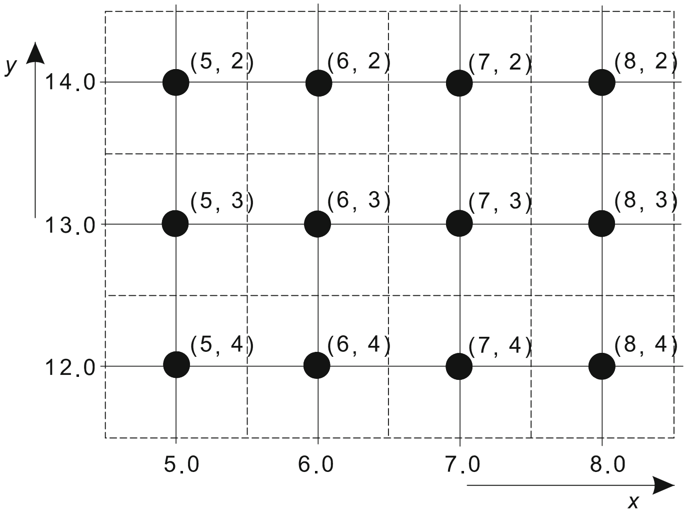


Fig. 1.4 Logical and device coordinates, based on $ymax = 16$

The dashed square around this dot denotes all points (x, y) satisfying

$$\begin{aligned} 7.5 &\le x < 8.5 \\ 13.5 &\le y < 14.5 \end{aligned}$$

All these points are converted to the pixel (8, 2) by our methods *iX* and *iY*.

Let us demonstrate this way of converting floating-point logical coordinates to integer device coordinates in a program that begins by drawing an equilateral triangle ABC, with the side AB at the bottom and point C at the top. Then, using

$$\begin{aligned} q &= 0.05 \\ p &= 1 - q = 0.95, \end{aligned}$$

we compute the new points A′, B′ and C′ near A, B and C and lying on the sides AB, BC and CA, respectively, writing:

```
xA1 = p * xA + q * xB;
yA1 = p * yA + q * yB;
xB1 = p * xB + q * xC;
yB1 = p * yB + q * yC;
xC1 = p * xC + q * xA;
yC1 = p * yC + q * yA;
```

We then draw the triangle A′B′C′, which is slightly smaller than ABC and turned a little counter-clockwise. Applying the same principle to triangle A′B′C′ to obtain a third triangle, A″B″C″, and so on, until 50 triangles have been drawn, the result will be as shown in Fig. 1.5.

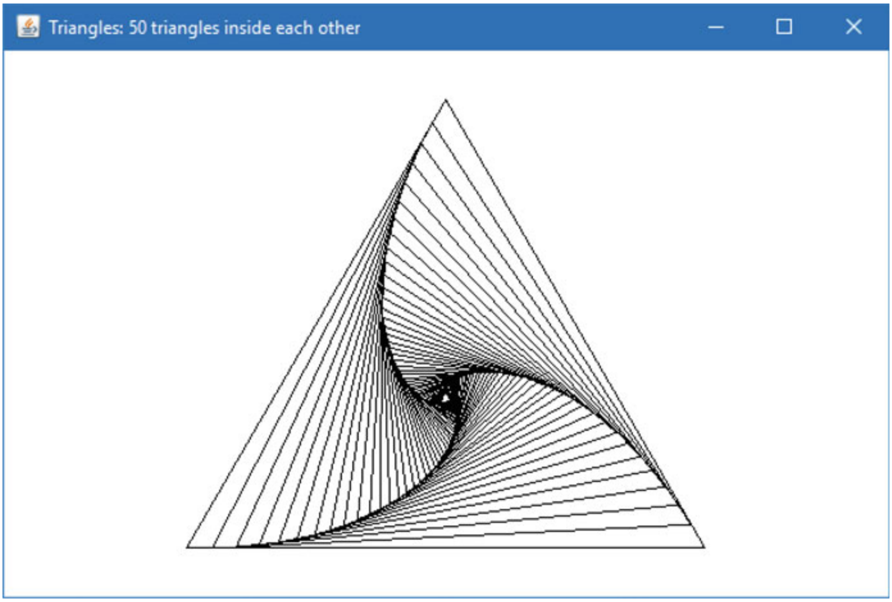


Fig. 1.5 Triangles, drawn inside each other

If we change the dimensions of the window, new equilateral triangles appear, again in the center of the canvas and with dimensions proportional to the size of this canvas. Without floating-point logical coordinates and with a *y*-axis pointing downward, this program would have been less easy to write:

```
// Triangles.java: This program draws 50 triangles inside
//                 each other.
import java.awt.*;
import java.awt.event.*;

public class Triangles extends Frame {
   public static void main(String[] args) {new Triangles();}

   Triangles() {
      super("Triangles: 50 triangles inside each other");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(600, 400);
      add("Center", new CvTriangles());
      setVisible(true);
   }
}

class CvTriangles extends Canvas {
   int maxX, maxY, minMaxXY, xCenter, yCenter;

   void initgr() {
      Dimension d = getSize();
      maxX = d.width - 1; maxY = d.height - 1;
      minMaxXY = Math.min(maxX, maxY);
      xCenter = maxX / 2; yCenter = maxY / 2;
   }

   int iX(float x) {return Math.round(x);}

   int iY(float y) {return maxY - Math.round(y);}

   public void paint(Graphics g) {
      initgr();
      float side = 0.95F * minMaxXY, sideHalf = 0.5F * side,
         h = sideHalf * (float) Math.sqrt(3),
         xA, yA, xB, yB, xC, yC, xA1, yA1, xB1, yB1, xC1, yC1, p, q;
      q = 0.05F; p = 1 - q;
      xA = xCenter - sideHalf; yA = yCenter - 0.5F * h;
```

```
        xB = xCenter + sideHalf; yB = yA;
        xC = xCenter; yC = yCenter + 0.5F * h;
        for (int i = 0; i < 50; i++) {
           g.drawLine(iX(xA), iY(yA), iX(xB), iY(yB));
           g.drawLine(iX(xB), iY(yB), iX(xC), iY(yC));
           g.drawLine(iX(xC), iY(yC), iX(xA), iY(yA));
           xA1 = p * xA + q * xB; yA1 = p * yA + q * yB;
           xB1 = p * xB + q * xC; yB1 = p * yB + q * yC;
           xC1 = p * xC + q * xA; yC1 = p * yC + q * yA;
           xA = xA1; xB = xB1; xC = xC1;
           yA = yA1; yB = yB1; yC = yC1;
        }
     }
  }
```

In the canvas class *CvTriangles* there is a method *initgr*. Together with the other program lines that precede the *paint* method in this class, *initgr* may also be useful in other programs.

It is important to notice that, on each triangle edge, the computed floating-point coordinates, not the integer device coordinates derived from them, are used for further computations. This principle, which applies to many graphics applications, can be depicted as follows:

$$\begin{array}{ccccccc}
\textit{float} & \rightarrow & \textit{float} & \rightarrow & \textit{float} & \rightarrow & \textit{float} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\textit{int} & & \textit{int} & & \textit{int} & & \textit{int}
\end{array}$$

which is in contrast to the following scheme, which we should avoid. Here *int* device coordinates containing rounding-off errors are used not only for graphics output but also for further computations, so that such errors will accumulate:

$$\begin{array}{ccccccc}
\textit{float} & & \textit{float} & & \textit{float} & & \textit{float} \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
\textit{int} & & \textit{int} & & \textit{int} & & \textit{int}
\end{array}$$

In summary, we compare and contrast the logical and device coordinate systems in the following table, in terms of (1) the convention used in the text, but not in Java programs, of this book, (2) the data types of the programming language, (3) the coordinate value domain, and (4) the direction of positive y-axis:

Coordinate system	Convention	Data type	Value domain	Positive *y*-axis
Logical	Lower-case letters	float	Continuous	Upward
Device	Upper-case letters	integer	Discrete	Downward

1.3 Anisotropic and Isotropic Mapping Modes

Mapping a Continuous Interval to a Sequence of Integers

Suppose we want to map an interval of real logical coordinates, such as

$$0 \leq x \leq 10.0$$

to the set of integer device coordinates {0, 1, 2, . . ., 9}. Unfortunately, the method:

```
int iX(float x){return Math.round(x);}
```

used in the previous section, is not suitable for this purpose because for any x greater than 9.5 (and not greater than 10) it returns 10, which does not belong to the allowed sequence 0, 1, ..., 9. In particular, it gives:

```
ix(10.0) = 10
```

while we want:

```
ix(10.0) = 9
```

This suggests that in an improved method *iX* we should use a multiplication factor of 0.9. We can also come to this conclusion by realizing that there are only nine small intervals between ten pixels labeled 0, 1, . . .9, as Fig. 1.6 illustrates. If we define the pixel width (= *pixelWidth*) as the distance between two successive pixels on a horizontal line, the above interval $0 \leq x \leq 10$ of logical coordinates (being real numbers) corresponds to $9 \times$ *pixelWidth*. So in this example we have:

$9 \times pixelWidth = 10.0$ (the length of the interval of logical coordinates)
$pixelWidth = 10/9 = 1.111\ldots$

In general, if a horizontal line of our canvas consists of n pixels, numbered 0, 1, . . ., *maxX* (where $maxX = n - 1$), and the corresponding (continuous) interval of logical coordinates is $0 \leq x \leq rWidth$, we can use the following method:

```
int iX(float x){return Math.round(x/pixelWidth);}
```

where *pixelWidth* is computed beforehand as follows:

```
maxX = n - 1;
pixelWidth = rWidth/maxX;
```

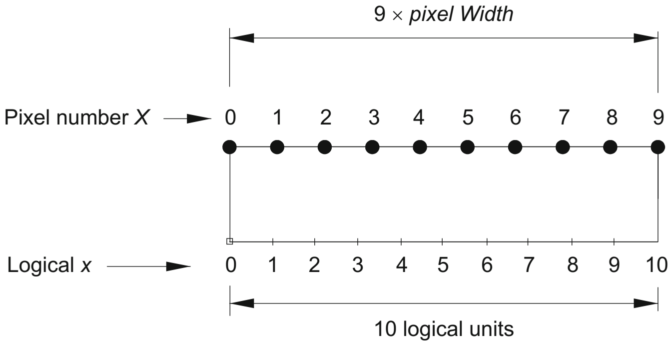


Fig. 1.6 Pixels lying 10/9 logical units apart

In the above example the integer n is equal to the interval length *rWidth*, but it is often desirable to use logical coordinates x and y satisfying

$$0 \leq x \leq rWidth$$
$$0 \leq y \leq rHeight$$

where *rWidth* and *rHeight* are real numbers, such as 10.0 and 7.5, respectively, which are quite different from the numbers of pixels that lie on horizontal and vertical lines. It will then be important to distinguish between *isotropic* and *anisotropic mapping modes*, as we will discuss in a moment.

As for the simpler method:

```
int iX(float x){return Math.round(x);}
```

of the previous section, this can be regarded as a special case of the improved one we have just seen, provided we use $pixelWidth = 1$, that is $rWidth = maxX$, or $rWidth = n - 1$. For example, if the drawing rectangle is 100 pixels wide, so that $n = 100$ and we can use the pixels $0, 1, 2, \ldots, 99 = maxX$ on a horizontal line, this simpler method *iX* works correctly if it is applied to logical x-coordinates satisfying

$$0 \leq x \leq rWidth = 99.0$$

The point to be noticed is that, due to the value $pixelWidth = 1$, the logical width is 99.0 here although the number of available pixels is 100.

Anisotropic Mapping Mode

The term *anisotropic mapping mode* implies that the scale factors for x and y are not necessarily equal, as the following code segment shows:

```
Dimension d = getSize();
maxX = d.width - 1; maxY = d.height - 1;
pixelWidth  = rWidth/maxX;
pixelHeight = rHeight/maxY;
...
int iX(float x){return Math.round(x/pixelWidth);}
int iY(float y){return maxY - Math.round(y/pixelHeight);}
float fx(int x){return x * pixelWidth;}
float fy(int y){return (maxY - y) * pixelHeight;}
```

We will use this in a demonstration program. Regardless of the window dimensions, the largest possible rectangle in this window has the logical dimensions 10.0 × 7.5. After clicking on a point of the canvas, the logical coordinates are shown as in Fig. 1.7.

Since there are no gaps between this largest possible rectangle and the window edges, we can only see this rectangle in Fig. 1.7 with some difficulty. In contrast, the screen will show this rectangle very clearly because we will make it red instead of black. Although the window dimensions in Fig. 1.7 have been altered by the user, the logical canvas dimensions are still 10.0 × 7.5. The text displayed in the window shows the coordinates of the point near the upper-right corner of the rectangle, as the cross-hair cursor indicates. If the user clicks exactly on that corner, the coordinate values 10.0 and 7.5 are displayed. This demonstration program is listed below.

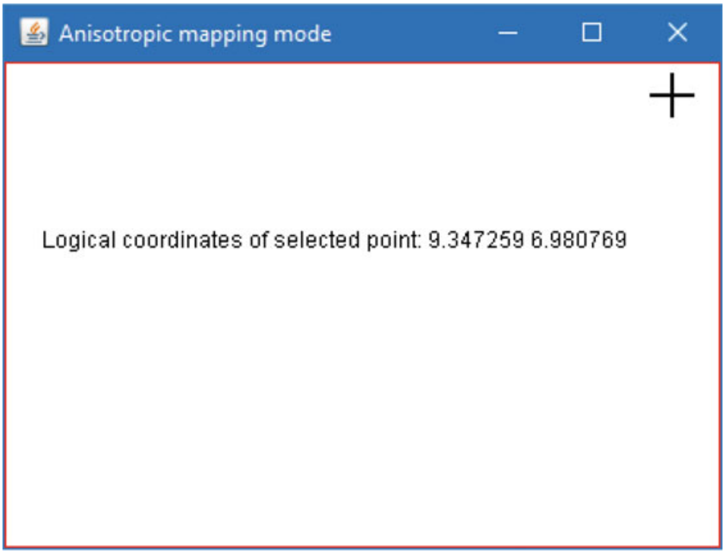


Fig. 1.7 Logical coordinates with anisotropic mapping mode

```
// Anisotr.java: The anisotropic mapping mode.
import java.awt.*;
import java.awt.event.*;

public class Anisotr extends Frame {
   public static void main(String[] args) {new Anisotr();}

   Anisotr() {
      super("Anisotropic mapping mode");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(400, 300);
      add("Center", new CvAnisotr());
      setCursor(Cursor.getPredefinedCursor(
         Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvAnisotr extends Canvas {
   int maxX, maxY;
   float pixelWidth, pixelHeight, rWidth = 10.0F, rHeight = 7.5F,
      xP = -1, yP;

   CvAnisotr() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            xP = fx(evt.getX()); yP = fy(evt.getY());
            repaint();
         }
      });
   }

   void initgr() {
      Dimension d = getSize();
      maxX = d.width - 1; maxY = d.height - 1;
      pixelWidth = rWidth / maxX; pixelHeight = rHeight / maxY;
   }

   int iX(float x) {return Math.round(x / pixelWidth);}
   int iY(float y) {return maxY - Math.round(y / pixelHeight);}
   float fx(int x) {return x * pixelWidth;}
   float fy(int y) {return (maxY - y) * pixelHeight;}
```

```
    public void paint(Graphics g) {
       initgr();
       int left = iX(0), right = iX(rWidth),
           bottom = iY(0), top = iY(rHeight);
       if (xP >= 0)
          g.drawString(
                "Logical coordinates of selected point: " +
                xP + " " + yP, 20, 100);
       g.setColor(Color.red);
       g.drawLine(left, bottom, right, bottom);
       g.drawLine(right, bottom, right, top);
       g.drawLine(right, top, left, top);
       g.drawLine(left, top, left, bottom);
    }
 }
```

With the anisotropic mapping mode, the actual length of a vertical unit can be different from that of a horizontal unit. This is the case in Fig. 1.7: although the rectangle is 10 units wide and 7.5 units high, its real height is less than 0.75 of its width. In particular, the anisotropic mapping mode is not suitable for drawing squares, circles and other shapes that require equal units in the horizontal and vertical directions.

Isotropic Mapping Mode

We can arrange for horizontal and vertical units to be equal in terms of their real size by using the same scale factor for x and y. Let us use the term *drawing rectangle* for the rectangle with dimensions *rWidth* and *rHeight*, in which we normally draw graphical output. Since these logical dimensions are constant, so is their ratio, which is not the case with that of the canvas dimensions. It follows that, with the isotropic mapping mode, the drawing rectangle will in general not be identical with the canvas. Depending on the current window size, either the top and bottom or the left and right edges of the drawing rectangle lie on those of the canvas.

Since it is normally desirable for a drawing to appear in the center of the canvas, it is often convenient with the isotropic mapping mode to place the origin of the logical coordinate system in that center. This implies that we will use the following logical-coordinate intervals:

$$-\tfrac{1}{2}rWidth \le x \le +\tfrac{1}{2}rWidth$$
$$-\tfrac{1}{2}rHeight \le y \le +\tfrac{1}{2}rHeight$$

Our methods *iX* and *iY* will map each logical coordinate pair (x, y) to a pair (X, Y) of device coordinates, where

$$X \in \{0, 1, 2, \ldots, maxX\}$$
$$Y \in \{0, 1, 2, \ldots, maxY\}$$

To obtain the same scale factor for x and y, we compute *rWidth*/*maxX* and *rHeight*/*maxY* and take the larger of these two values. This maximum value, *pixelSize*, is then used in the methods *iX* and *iY*, as shown below:

```
Dimension d = getSize();
int maxX = d.width - 1, maxY = d.height - 1;
pixelSize = Math.max(rWidth/maxX, rHeight/maxY);
centerX = maxX/2; centerY = maxY/2;
...
int iX(float x){return Math.round(centerX + x/pixelSize);}
int iY(float y){return Math.round(centerY - y/pixelSize);}
float fx(int x){return (x - centerX) * pixelSize;}
float fy(int y){return (centerY - y) * pixelSize;}
```

We will use this code in a program that draws a square, two corners of which touch either the midpoints of the horizontal canvas edges or those of the vertical ones. It also displays the coordinates of a point on which the user clicks, as the left window of Fig. 1.8 shows.

In this illustration, we pay special attention to the corners of the drawn square that touch the boundaries of the drawing rectangle. These corners do not lie on the window frame, but just inside it. For the square on the left we have:

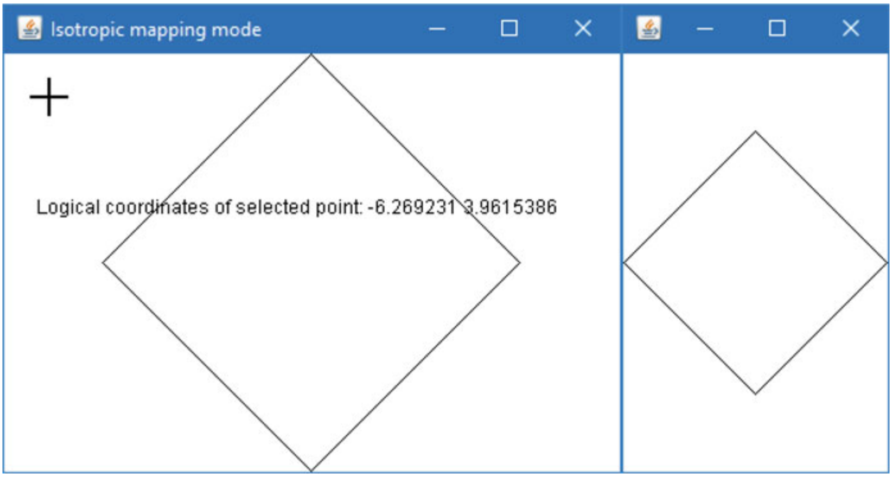


Fig. 1.8 Windows after changing their sizes

Figure 1.8, left	Logical coordinate y	Device coordinate $iY(y)$
Top corner	$+rHeight/2$	0
Bottom corner	$-rHeight/2$	$maxY$

By contrast, with the square that has been drawn in the narrow window on the right, it is the corners on the left and the right that lie just within the frame:

Figure 1.8, right	Logical coordinate x	Device coordinate $iX(x)$
Left corner	$-rWidth/2$	0
Right corner	$+rWidth/2$	$maxX$

The following program uses a drawing rectangle with logical dimensions $rWidth = rHeight = 10.0$. If we replaced this value 10.0 with any other positive constant, the output would be the same.

```
// Isotrop.java: The isotropic mapping mode.
//    Origin of logical coordinate system in canvas
//    center; positive y-axis upward.
//    Square (turned 45 degrees) just fits into canvas.
//    Mouse click displays logical coordinates of
//    selected point.
import java.awt.*;
import java.awt.event.*;

public class Isotrop extends Frame {
   public static void main(String[] args) {new Isotrop();}

   Isotrop() {
      super("Isotropic mapping mode");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(400, 300);
      add("Center", new CvIsotrop());
      setCursor(Cursor.getPredefinedCursor(
         Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvIsotrop extends Canvas {
   int centerX, centerY;
   float pixelSize, rWidth = 10.0F, rHeight = 10.0F,
      xP = 1000000, yP;
```

```
   CvIsotrop() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            xP = fx(evt.getX()); yP = fy(evt.getY());
            repaint();
         }
      });
   }

   void initgr() {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
   }

   int iX(float x) {return Math.round(centerX + x / pixelSize);}
   int iY(float y) {return Math.round(centerY - y / pixelSize);}
   float fx(int x) {return (x - centerX) * pixelSize;}
   float fy(int y) {return (centerY - y) * pixelSize;}

   public void paint(Graphics g) {
      initgr();
      int left = iX(-rWidth / 2), right = iX(rWidth / 2),
          bottom = iY(-rHeight / 2), top = iY(rHeight / 2),
          xMiddle = iX(0), yMiddle = iY(0);
      g.drawLine(xMiddle, bottom, right, yMiddle);
      g.drawLine(right, yMiddle, xMiddle, top);
      g.drawLine(xMiddle, top, left, yMiddle);
      g.drawLine(left, yMiddle, xMiddle, bottom);
      if (xP != 1000000)
         g.drawString("Logical coordinates of selected point: " + xP
               + " " + yP, 20, 100);
   }
}
```

1.4 Defining a Polygon Through Mouse Interaction

We will use the conversion methods of the previous program in a more interesting application, which enables the user to define a polygon by clicking on points to indicate the positions of the vertices. Figure 1.9 shows such a polygon, with twenty vertices, labeled 0, 1, 2, . . ., 19. The user has defined these vertices by clicking at their positions. The first vertex, labeled 0, shows a tiny square. To indicate that the

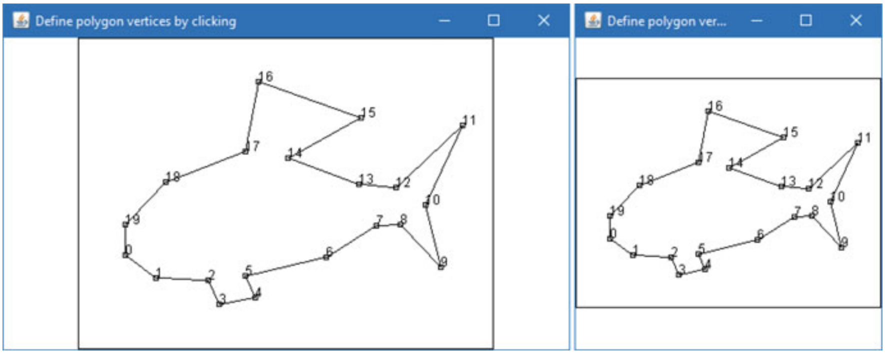


Fig. 1.9 Polygon defined by user

polygon is complete, the user can either click on vertex 0 again in the tiny square, or use the right mouse button to define the last vertex, 19 in this example; in the latter case vertex 19 is automatically connected with vertex 0.

The large rectangle surrounding the polygon is the drawing rectangle: only vertices inside this rectangle are guaranteed to appear again if the user changes the dimensions of the window. As usual, the user can change the dimensions of the window by dragging its edges or corners. It is then desirable that the picture in the window stays reasonably centered in the window, that it will not be truncated when the window gets too small for its original dimensions, and that its *aspect ratio* (ratio width: height) will not change. Figure 1.9 shows that these requirements are met in the program *DefPoly.java* and in particular in its canvas class *CvDefPoly*. We now summarize the requirements for this program:

- The first vertex is drawn as a tiny square.
- If a later vertex is inside the tiny square, the drawing of one polygon is complete. Alternatively, clicking a point with the right mouse button makes this point the final vertex of the polygon.
- Only vertices in the drawing rectangle are drawn.
- The drawing rectangle (see Fig. 1.9 left and right) is either as high or as wide as the window, yet maintaining its height/width ratio regardless of the window shape.
- When the user changes the shape of the window, the size of the drawn polygon changes in the same way as that of the drawing rectangle, as does the surrounding white space of the polygon.

We will use the isotropic mapping mode to implement this program, and use a data structure called *vertex vector* to store the vertices of the polygon to be drawn. We then design the program with the following algorithmic steps:

1. Activate the mouse;
2. When the left mouse button is pressed

2.1. Get x- and y-coordinates at where the mouse is clicked;
2.2. If it is the first vertex

Then empty vertex vector;
Else If the vertex is inside the tiny square (the first vertex)

Then finish the current polygon;
Else store this vertex in vertex vector (i.e. not last vertex);

3. When the right mouse button is pressed

3.1. Get x- and y-coordinates at where the mouse is clicked;
3.2. Store this vertex in vertex vector; it is the last vertex;
3.3. Finish the current polygon;

4. Draw the polygon using the vertices in vertex vector.

The last step to draw all the polygon vertices can be detailed as follows:

1. Obtain the dimensions of the drawing rectangle based on logical coordinates;
2. Draw the drawing rectangle;
3. Get the first vertex from vertex vector;
4. Draw a tiny square at the vertex location;
5. Draw a line between every two consecutive vertices stored in vertex vector.

In both the left and the right windows of Fig. 1.9, despite their different dimensions, the drawing rectangle has a width of 10 and a height of 7.5 logical units, as the canvas class *CvDefPoly* of the following program shows:

```
// DefPoly.java: Drawing a polygon.
// Uses: CvDefPoly (discussed below).
import java.awt.*;
import java.awt.event.*;

public class DefPoly extends Frame {
   public static void main(String[] args) {new DefPoly();}

   DefPoly() {
      super("Define polygon vertices by clicking");
      addWindowListener(new WindowAdapter() {
        public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(500, 300);
      add("Center", new CvDefPoly());
      setCursor(Cursor.getPredefinedCursor(Cursor.
      CROSSHAIR_CURSOR));
      setVisible(true);
    }
}
```

The class *CvDefPoly*, used in this program, is listed below. We define this class in a separate file, *CvDefPoly.java*, so that it is easier to use elsewhere, as we will see in Sect. 2.6:

```
// CvDefPoly.java: To be used in other program files.
// A class that enables the user to define
// a polygon by clicking the mouse.
// Uses: Point2D (discussed below).
import java.awt.*;
import java.awt.event.*;
import java.util.*;

class CvDefPoly extends Canvas {
   Vector v = new Vector();
   float x0, y0, rWidth = 10.0F, rHeight = 7.5F, pixelSize;
   boolean ready = true;
   int centerX, centerY;

   CvDefPoly() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            float xA = fx(evt.getX()), yA = fy(evt.getY());
            if (ready) {
               v.removeAllElements();
               x0 = xA; y0 = yA;
               ready = false;
            }
            float dx = xA - x0, dy = yA - y0;
            if (v.size() > 0 &&
               dx * dx + dy * dy < 20 * pixelSize * pixelSize)
               ready = true;
            else
               v.addElement(new Point2D(xA, yA));

            // Right mouse button indicates the final vertex:
            if(evt.getModifiers()==InputEvent.BUTTON3_MASK) {
               ready = true;
            }
            repaint();
         }
      });
   }
```

```
   void initgr() {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
   }

   int iX(float x) {return Math.round(centerX + x / pixelSize);}
   int iY(float y) {return Math.round(centerY - y / pixelSize);}
   float fx(int x) {return (x - centerX) * pixelSize;}
   float fy(int y) {return (centerY - y) * pixelSize;}

   public void paint(Graphics g) {
      initgr();
      int left = iX(-rWidth / 2), right = iX(rWidth / 2),
          bottom = iY(-rHeight / 2), top = iY(rHeight / 2);
      g.drawRect(left, top, right - left, bottom - top);
      int n = v.size();
      if (n == 0)
         return;
      Point2D a = (Point2D) (v.elementAt(0));
      // Show tiny rectangle around first vertex:
      g.drawRect(iX(a.x) - 2, iY(a.y) - 2, 4, 4);
      for (int i = 1; i <= n; i++) {
         if (i == n && !ready)
            break;
         Point2D b = (Point2D) (v.elementAt(i % n));
         g.drawLine(iX(a.x), iY(a.y), iX(b.x), iY(b.y));
         g.drawRect(iX(b.x) - 2, iY(b.y) - 2, 4, 4);
            // Tiny rectangle
         a = b;
         g.drawString(""+(i%n), iX(b.x), iY(b.y));
            // Display vertex number as text.
      }
   }
}
```

The class *Point2D*, used in the above file, will also be useful in other programs, so that we define this in another separate file, *Point2D.java*:

```
// Point2D.java: Class for points in logical coordinates.
class Point2D {
  float x, y;
   Point2D(float x, float y){this.x = x; this.y = y;}
}
```

After a complete polygon has been shown (which is the case when the user has revisited the first vertex), the user can once again click a point. The polygon then disappears and that point will then be the first vertex of a new polygon.

Note that the comment line:

```
// Uses: CvDefPoly (discussed below).
```

occurring in the file *DefPoly.java*, does *not* imply that adding the class *CvDefPoly* is sufficient. It is meant to refer to the file *CvDefPoly.java*. All classes defined in that file are required and so are the classes that are subsequently referred to. Since.

```
// Uses: Point2D (discussed below).
```

occurs in the file *CvDefPoly.java*, the program *DefPoly.java* also requires the class *Point2D*. Comments such as those above are very helpful if different directories are used, for example, one for each chapter. However, since class names are unique throughout this book, it is possible to place all program files in the same directory. In this way, each required class will be available.

Exercises

1.1 The calls

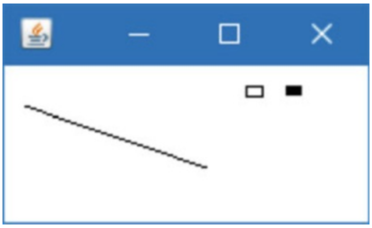
```
g.drawLine(10, 20, 100, 50);
g.drawRect(120, 10, 8, 5);
g.fillRect(140, 10, 8, 5);
```

give results as shown in Fig. 1.10. How many pixels are put on the screen by each of them?

1.2 Replace the triangles of program *Triangles.java* with squares and draw a great many of them, arranged in a chessboard, as shown in Fig. 1.11.

As usual, this chessboard consists of $n \times n$ normal squares (with horizontal and vertical edges), where $n = 8$. Each of these actually consists of k squares of different sizes, with $k = 10$. Finally, the value $q = 0.2$ (and $p = 1 - q = 0.8$)

Fig. 1.10 Line, drawn and filled rectangles



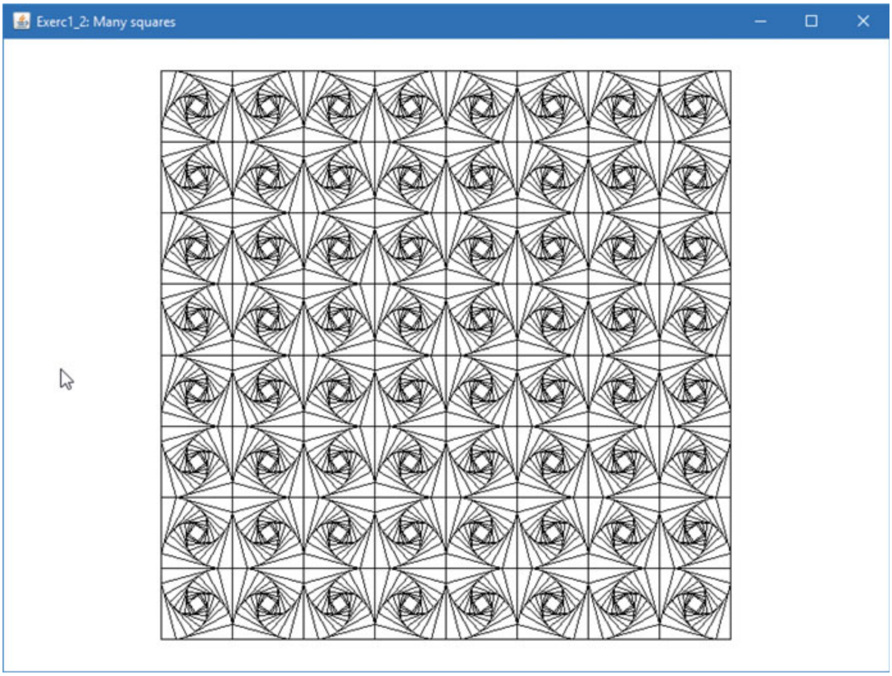


Fig. 1.11 A chessboard of squares

was used to divide each edge into two parts with ratio $p : q$ (see also program *Triangles.java* of Sect. 1.2), but the interesting pattern of Fig. 1.11 was obtained by reversing the roles of p and q in half of the $n \times n$ 'normal' squares, which is similar to the black and whites squares of a normal chessboard. Your program should accept the values n, k and q as program arguments.

1.3 Draw a set of concentric pairs of squares, each consisting of a square with horizontal and vertical edges and one rotated through 45°. Except for the outermost square, the vertices of each square are the midpoints of the edges of its immediately surrounding square, as Fig. 1.12 shows. It is required that all lines are exactly straight, and that vertices of smaller squares lie exactly on the edges of larger ones.

1.4 Write a program that draws a pattern of hexagons, as shown in Fig. 1.13. The vertices of a (regular) hexagon lie on its so-called circumscribed circle. The user must be able to specify the radius of this circle by clicking a point near the upper-left corner of the drawing rectangle. Then the distance between that point and that corner is to be used as the radius of the circle just mentioned. There must be as many hexagons of the specified size as possible and the margins on the left and the right must be equal. The same applies to the upper and lower margins, as Fig. 1.13 shows.

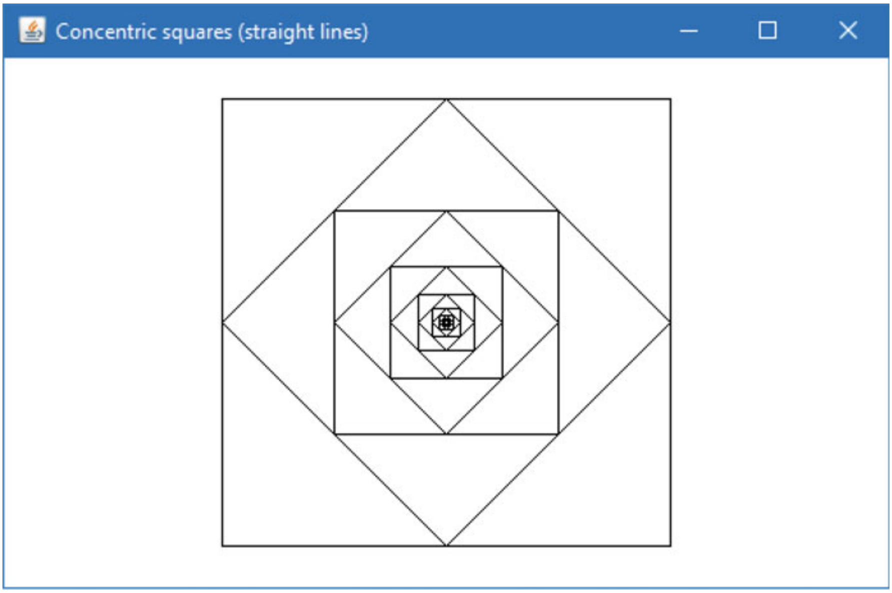


Fig. 1.12 Concentric squares

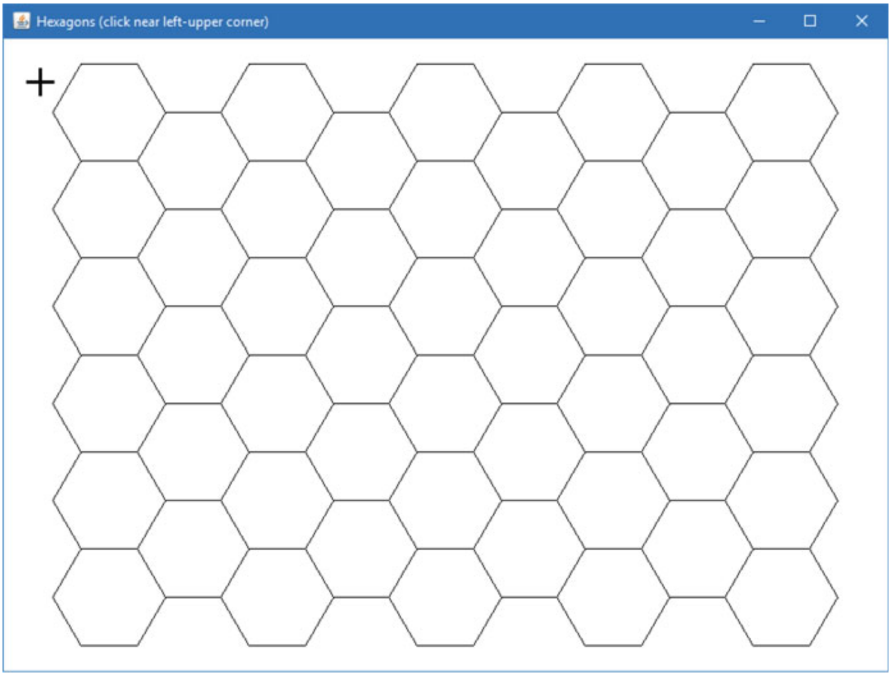


Fig. 1.13 Hexagons

1.5 Write a class *Lines* containing a static method *dashedLine* to draw dashed lines, in such a way that we can write

```
Lines.dashedLine(g, xA, yA, xB, yB, dashLength);
```

where *g* is a variable of type *Graphics*, *xA*, *yA*, *xB*, *yB* are the device coordinates of the endpoints A and B, and *dashLength* is the desired length (in device coordinates) of a single dash. There should be a dash, not a gap, at each endpoint of a dashed line. Figure 1.14 shows eight dashed lines drawn in this way, with $dashLength = 20$.

1.6 Write a program to draw the interface of the Game of Tetris, as shown in Fig. 1.15. The game involves playing with different shapes, each composed of four squares. The main area (large rectangle) should be sized as 10×20 squares (i.e. 10 squares wide and 20 squares high). The small rectangle on the right shows the "next shape" that will soon appear in the main area (a red "L" shape in the example). If you have not played Tetris before, you may find useful information on the Web (e.g. http://en.wikipedia.org/wiki/Tetris).

Your tasks include:

- Draw everything shown in this figure (the position of each component, e.g. rectangle, does not have to be exactly the same as in the figure).
- If the mouse cursor moves inside the main area, "PAUSE" (in a large font) will be displayed; and if the cursor moves out of the area, "PAUSE" will disappear.

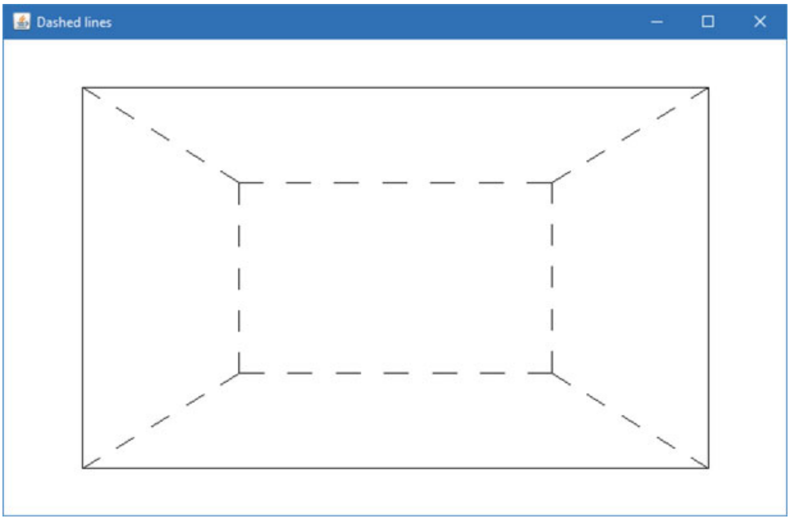


Fig. 1.14 Dashed lines

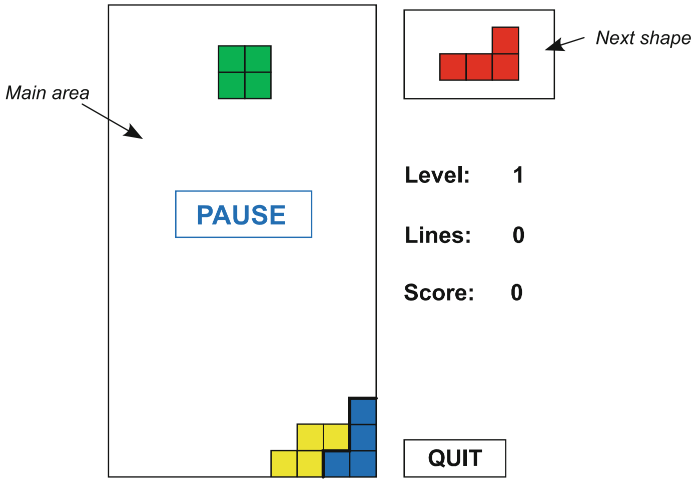


Fig. 1.15 The Tetris interface

- Change of the window dimension will only possibly change the size, but not the relative position and aspect ratio of any component.
- If the button "QUIT" is pressed, the program terminates and quits (this should not be the quit from the window's standard pull-down menu).

Chapter 2
Applied Geometry

Before proceeding with specific computer graphics subjects, we will discuss related mathematics, which will be frequently used in this book. You can skip the first two sections of this chapter if you are familiar with vectors. After this general part, we will discuss several useful algorithms, needed for the exercises at the end of this chapter and for the topics in the later chapters. For example, Chap. 6 will present approaches on how to handle polygons that are the faces of 3D solid objects. Since polygons in general are difficult to handle, we will divide them into triangles, as discussed in Sect. 2.6.

2.1 Vectors

We begin with the mathematical notion of a vector, which should not be confused with the standard class *Vector*, available in Java to store an arbitrary number of objects. Recall that we have used this *Vector* class in Sect. 1.4 to store polygon vertices.

A *vector* is a directed line segment, characterized by its length and its direction only. Figure 2.1 shows two representations of the same vector $\mathbf{a} = \mathbf{PQ} = \mathbf{b} = \mathbf{RS}$. Thus a vector is not altered by a translation (or shift as a non-technical term).

The sum $\mathbf{c}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$, written

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

can be obtained as the diagonal of a parallelogram, with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ starting at the same point, as shown in Fig. 2.2.

The length of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$. A vector with zero length is the zero vector, written $\mathbf{0}$. The notation $-\mathbf{a}$ is used for the vector that has length $|\mathbf{a}|$ and whose direction is opposite to that of $\mathbf{a}$. For any vector $\mathbf{a}$ and real number c, the vector $c\mathbf{a}$ has length $|c||\mathbf{a}|$. If $\mathbf{a} = \mathbf{0}$ or $c = 0$, then $c\mathbf{a} = \mathbf{0}$; otherwise $c\mathbf{a}$ has the direction of $\mathbf{a}$ if

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2_2

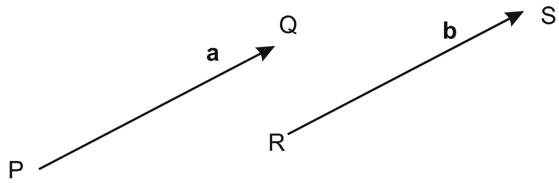


Fig. 2.1 Two equal vectors

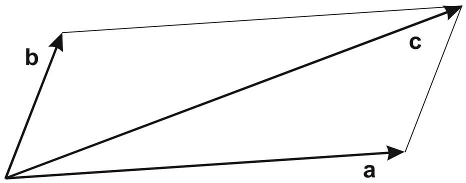


Fig. 2.2 Vector addition

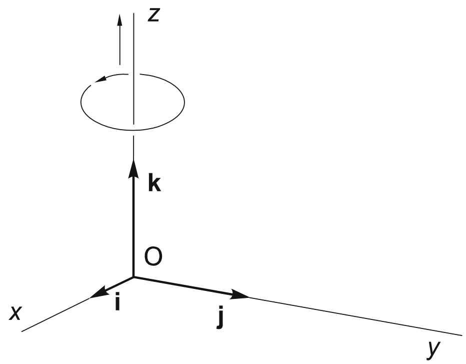


Fig. 2.3 Right-handed coordinate system

$c > 0$ and the opposite direction if $c < 0$. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and real numbers c, k, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$
$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$
$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$
$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$
$$(c + k)\mathbf{u} = c\mathbf{u} + k\mathbf{u}$$
$$c(k\mathbf{u}) = (ck)\mathbf{u}$$
$$1\mathbf{u} = \mathbf{u}$$
$$0\mathbf{u} = \mathbf{0}$$

Figure 2.3 shows three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in a 3-dimensional space. They are mutually perpendicular and have length 1. Their directions are the positive directions of the coordinate axes. We say that **i**, **j** and **k** form a triple of orthogonal *unit vectors*. The coordinate system is right-handed, which means that if a rotation of **i** in the direction of **j** through 90° corresponds to turning a right-handed screw, then **k** has the direction in which the screw advances.

We often choose the origin O of the coordinate system as the initial point of all vectors. Any vector **v** can be written as a *linear combination* of the unit vectors **i**, **j**, and **k**:

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The real numbers x, y and z are the coordinates of the endpoint P of vector $\mathbf{v} = \mathbf{OP}$. We often write this vector **v** as

$$\mathbf{v} = [x \quad y \quad z] \quad \text{or as} \quad \mathbf{v} = (x, y, z)$$

The numbers x, y, z are sometimes called the *elements* or *components* of vector **v**.

The possibility of writing a vector as a sequence of coordinates, such as (x, y, z) for three-dimensional space, has led to the use of this term for sequences in general. This explains the name *Vector* for a standard Java class. The only aspect this *Vector* class has in common with the mathematical notion of vector is that both are related to sequences.

2.2 Inner Product and Vector Product

Inner Product

The *inner product* or *dot product* of two vectors **a** and **b** is a real number, written as $\mathbf{a} \cdot \mathbf{b}$ and defined as

$$\begin{array}{ll} \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma & \text{if } \mathbf{a} \neq 0 \text{ and } \mathbf{b} \neq 0 \\ \mathbf{a} \cdot \mathbf{b} = 0 & \text{if } \mathbf{a} = 0 \text{ or } \mathbf{b} = 0 \end{array} \tag{2.1}$$

where γ is the angle between **a** and **b**. It follows from the first equation that $\mathbf{a} \cdot \mathbf{b}$ is also zero if $\gamma = 90°$. Applying this definition to the unit vectors **i**, **j** and **k**, we find

$$\begin{array}{l} \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = 0 \end{array} \tag{2.2}$$

Setting $\mathbf{b} = \mathbf{a}$ in Eq. (2.1), we have $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, so

$$|\mathbf{a}| = \sqrt{|\mathbf{a} \cdot \mathbf{a}|}$$

Some important properties of inner products are

$$c(k\mathbf{u}\cdot\mathbf{v}) = ck(\mathbf{u}\cdot\mathbf{v})$$
$$(c\mathbf{u}+k\mathbf{v})\cdot\mathbf{w} = c\mathbf{u}\cdot\mathbf{w}+k\mathbf{v}\cdot\mathbf{w}$$
$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$$
$$\mathbf{u}\cdot\mathbf{u} = 0 \text{ only if } \mathbf{u} = \mathbf{0}$$

The inner product of two vectors $\mathbf{u} = [u_1\ u_2\ u_3]$ and $\mathbf{v} = [v_1\ v_2\ v_3]$ can be computed as

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

We can prove this by developing the right-hand side of the following equality as the sum of nine inner products and then applying Eq. (2.2):

$$\mathbf{u}\cdot\mathbf{v} = (u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\cdot(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})$$

Vector Product

The *vector product* or *cross product* of two vectors **a** and **b** is written

$$\mathbf{a}\times\mathbf{b}$$

and is a vector **v** with the following properties. If $\mathbf{a} = c\mathbf{b}$ for some scalar c, then $\mathbf{v} = \mathbf{0}$. Otherwise the length of **v** is equal to

$$|\mathbf{v}| = |\mathbf{a}|\,|\mathbf{b}|\sin\gamma$$

where γ is the angle between **a** and **b**, and the direction of **v** is perpendicular to both **a** and **b** and is such that **a**, **b** and **v**, in that order, form a right-handed triple. This means that if **a** is rotated through an angle $\gamma < 180°$ in the direction of **b**, then **v** has the direction of the advancement of a right-handed screw if turned in the same way. Note that the length $|\mathbf{v}|$ is equal to the area of a parallelogram of which the vectors **a** and **b** can act as edges, as Fig. 2.4 shows.

The following properties of vector products follow from this definition:

$$\begin{array}{lcll}
(k\mathbf{a})\times\mathbf{b} & = & k(\mathbf{a}\times\mathbf{b}) & \text{for any real number } k\\
\mathbf{a}\times(\mathbf{b}+\mathbf{c}) & = & \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} & \\
\mathbf{a}\times\mathbf{b} & = & -\mathbf{b}\times\mathbf{a} &
\end{array}$$

In general, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. Applying our definition of vector product to the unit vectors **i**, **j**, **k** (see Fig. 2.3), we have

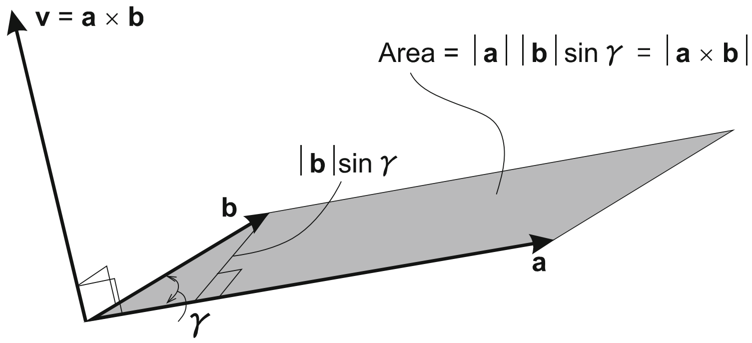


Fig. 2.4 Vector product **a** × **b**

$$\begin{array}{lll} \mathbf{i}\times\mathbf{i}=\mathbf{0} & \mathbf{j}\times\mathbf{j}=\mathbf{0} & \mathbf{k}\times\mathbf{k}=\mathbf{0} \\ \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{j}\times\mathbf{k}=\mathbf{i} & \mathbf{k}\times\mathbf{i}=\mathbf{j} \\ \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \end{array}$$

Using these vector products in the expansion of

$$\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

we obtain

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

which can be written as

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i}+\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{j}+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

We write this in a form that is easy to remember:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This is a mnemonic aid rather than a true determinant, since the elements of the first row are vectors instead of numbers.

If **a** and **b** are neighboring sides of a parallelogram, as shown in Fig. 2.4, the area of this parallelogram is the length of vector **a** × **b**. This follows from our definition of vector product, according to which $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\gamma$ is the length of vector **a** × **b**.

2.3 The Orientation of Three Points

We will now introduce a concept that will be very useful in three-dimensional computer graphics. Suppose that we are given an ordered triple (A, B, C) of three points in the xy-plane and we want to know their orientation; in other words, we wish to know whether we turn counter-clockwise or clockwise when visiting these points in the given order. Figure 2.5 shows the possibilities, which we also refer to as *positive* and *negative* orientations, respectively.

There is a third possibility, namely that the points A, B and C lie on a straight line. We will consider the orientation to be zero in this case. If we plot the points on paper, we see immediately which of these three cases applies, but we now need a means to find the orientation by computation, using only the given coordinates x_A, y_A, x_B, y_B, x_C, y_C.

Let us define the two vectors $\mathbf{a} = \mathbf{CA}$ and $\mathbf{b} = \mathbf{CB}$, as shown in Fig. 2.6. Clearly, the orientation of the original points A, B and C is positive if we can turn the vector $\mathbf{a}$ counter-clockwise through a positive angle less than 180° to obtain the direction of the vector $\mathbf{b}$. Since vectors are determined by their directions and lengths only,

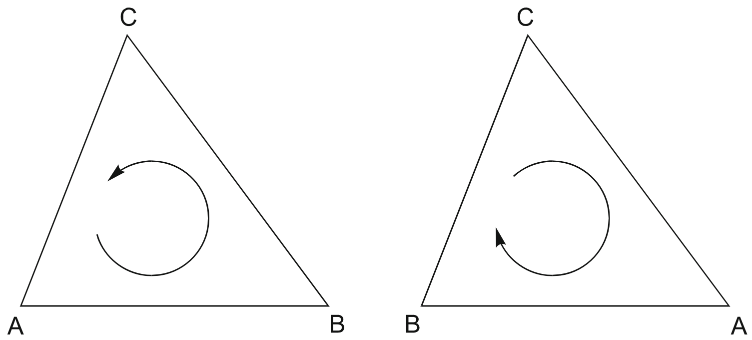
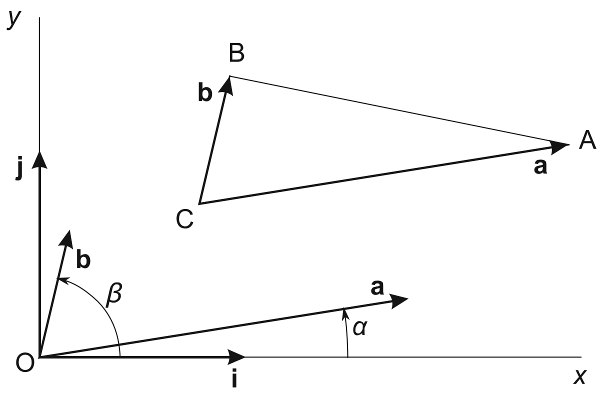


Fig. 2.5 Counter-clockwise (*left*) and clockwise (*right*) orientation of A, B, C

Fig. 2.6 Using vectors **a** and **b** instead of triangle edges CA and CB



not their locations, we may let them start at the origin O instead of at point C, as Fig. 2.6 shows. Although this orientation problem is essentially two-dimensional, and can be solved using only 2D concepts, as we will see in a moment, it is convenient to use 3D space. As usual, the unit vectors **i**, **j** and **k** have the directions of the positive x-, y- and z-axes. In Fig. 2.6, we imagine the vector **k**, like **i** and **j** starting at O, and pointing towards us. Denoting the endpoints of the translated vectors **a** and **b**, starting at O, by $(a_1, a_2, 0)$ and $(b_1, b_2, 0)$, we have

$$\mathbf{a} = a_1 i + a_2 j + 0\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + 0\mathbf{k}$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} = (a_1 b_2 - a_2 b_1)\mathbf{k}$$

This expresses the fact that $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the xy-plane and either in the same direction as

$$\mathbf{k} = \mathbf{i} \times \mathbf{j}$$

or in the opposite direction, depending on the sign of $a_1 b_2 - a_2 b_1$. If this expression is positive, the relationship between **a** and **b** is similar to that of **i** and **j**: we can turn **a** counter-clockwise through an angle less than 180° to obtain the direction of **b**, in the same way as we can do this with **i** to obtain **j**. In general, we have

$$a_1 b_2 - a_2 b_1 \begin{cases} > 0 : \text{orientation of A, B and C positive (counter-clockwise)} \\ = 0 : \text{A, B and C lie on the same line} \\ < 0 : \text{orientation of A, B and C negative (clockwise)} \end{cases}$$

An Alternative, Two-Dimensional Solution

It would be unsatisfactory if we were unable to solve the above orientation problem by using only two-dimensional concepts. To provide such an alternative solution, we use the angles α between vector **a** and the positive x-axis and β between **b** and this axis (see Fig. 2.6). Then the orientation we are interested in depends upon the angle $\beta - \alpha$. If this angle lies between 0 and π, the orientation is clearly positive, but it is negative if this angle lies between π and 2π (or between $-\pi$ and 0). We can express this by saying that the orientation in question depends on the value of sin $(\beta - \alpha)$ rather than on the angle $\beta - \alpha$ itself. More specifically, the orientation has the same sign as

$$\sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha = \frac{b_2}{|\mathbf{b}|}\frac{a_1}{|\mathbf{a}|} - \frac{b_1}{|\mathbf{b}|}\frac{a_2}{|\mathbf{a}|} = \frac{a_1b_2 - a_2b_1}{|\mathbf{a}|\,|\mathbf{b}|}$$

Since the denominator in this expression is the product of two vector lengths, it is positive, so that we have again found that the orientation of A, B and C and $a_1b_2 - a_2b_1$ have the same sign. Due to unfamiliarity with the above trigonometric formula, some readers may find the former, more visual 3D approach easier to remember.

A Useful Java Method

We will often use the method *area*2, listed in the Java class *Tools2D* shown below. This method is based on the results we have found. It takes three arguments of class *Point*2*D*, discussed at the end of Chap. 1. Note that, in accordance with Java convention, we use lower-case variable names *a*, *b*, *c* for the points A, B and C:

```
// Tools2D.java: Class to be used in other program files.
// Uses: Point2D (Section 1.4).
import java.util.Vector;

class Tools2D {
   static float area2(Point2D a, Point2D b, Point2D c) {
      return (a.x - c.x) * (b.y - c.y) -
          (a.y - c.y) * (b.x - c.x);
   }

   static float distance2(Point2D p, Point2D q) {
      float dx = p.x - q.x, dy = p.y - q.y;
      return dx * dx + dy * dy;
   }

   static boolean insideTriangle(Point2D a, Point2D b, Point2D c,
         Point2D p){ // ABC is assumed to be counter-clockwise
      return area2(a, b, p) >= 0 &&
             area2(b, c, p) >= 0 &&
             area2(c, a, p) >= 0;
   }
}
```

As we will see in Sect. 2.4, this method *area2* computes the area of the triangle ABC multiplied by 2, or, if A, B and C are clockwise, by −2. If we are interested only in the orientation of the points A, B and C, each of type *Point*2D, we can write:

```
if (Tools2D.area2(a, b, c) > 0) {
   ... // A, B and C counter-clockwise
}
else {
   ...
   // A, B and C clockwise (unless the area2 method return 0;
   // in that case A, B and C lie on the same line).
}
```

We will discuss the method *insideTriangle* of this class in Sect. 2.5. The method *distance2* computes the square of the distance between two points. In many applications we can use this square as well as the distance itself, saving the square root computation. For example, to test whether point A or point B is nearer to point P, we may write.

```
if (Tools2D.distance2(a, p) < Tools2D.distance2(b, p))
   ...
```

Obviously, we would need to use the *sqrt* method to compute the distance between A and P, writing, for example,

```
float d = (float)Math.sqrt(Tools2D.distance2(a, p));
```

2.4 Polygons and Their Areas

A *polygon* is a sequence $P_0, P_1, \ldots, P_{n-1}$ of vertices, where $n \geq 3$, with associated edges $P_0P_1, P_1P_2, \ldots, P_{n-2}P_{n-1}, P_{n-1}P_0$. In this book we will restrict ourselves to simple polygons, that is, polygons of which nonadjacent edges do not intersect. Figure 2.7 shows two simple and two non-simple polygons, for all of which $n = 5$.

Besides non-simple polygons, such as those in Fig. 2.7c, d, we usually also ignore polygons in which three successive vertices lie on the same line. A vertex of

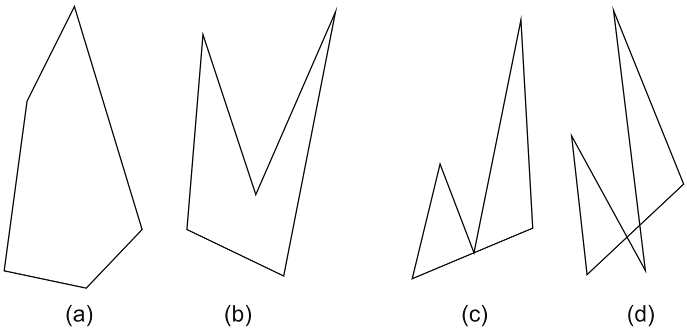


Fig. 2.7 Two simple and two non-simple polygons

a polygon is said to be *convex* if the interior angle between the two edges meeting at that vertex is less than 180°. If all vertices of a polygon are convex, the polygon itself is said to be convex, as is the case with Fig. 2.7a. Non-convex vertices are referred to as *concave* or *reflex*. If a polygon has at least one concave vertex, the polygon is said to be *concave*. Figure 2.7b shows a concave polygon because the vertex in the middle is concave. All triangles are convex, and each polygon has at least three convex vertices.

If we are given the vertices $P_0, P_1, \ldots, P_{n-1}$ of a polygon, it may be desirable to determine if this vertex sequence is counter-clockwise. If we know that the second vertex, P_1, is convex, we can simply write

```
if (Tools2D.area2(p[0], p[1], p[2]) > 0)
   ... // Counter-clockwise
else
   ... // Clockwise
```

The problem is more interesting if no information about any convex vertex is available. We then have to detect such a vertex. This is an easy task, since any vertex whose *x*- or *y*-coordinate is extreme is convex. For example, we can use a vertex whose *x*-coordinate is not greater than that of any other vertex. If we do not want to exclude the case of three successive vertices lying on the same line, we must pay special attention to the case of three such vertices having the minimum *x*-coordinate. Therefore, among all vertices with an *x*-coordinate equal to this minimum value, we choose the lowest one, that is, the one with the least *y*-coordinate. The following method is based on this idea:

```
static boolean ccw(Point2D[] p) {
   int n = p.length, k = 0;
   for (int i=1; i<n; i++)
      if (p[i].x <= p[k].x &&
          (p[i].x < p[k].x || p[i].y < p[k].y))
          k = i;
   // p[k] is a convex vertex.
   int prev = k - 1, next = k + 1;
   if (prev == -1) prev = n - 1;
   if (next == n) next = 0;
   return Tools2D.area2(p[prev], p[k], p[next]) > 0;
}
```

We should be aware that one very strange situation is still possible: all *n* vertices may lie on the same line. In that case, the method *ccw* will return the value *false*.

We will use this method in Sect. 2.6, in which program *PolyTria.java* will divide a user-provided polygon into triangles.

The Area of a Polygon

As we have seen in Fig. 2.4, the cross product **a** × **b** is a vector whose length is equal to the area of a parallelogram of which **a** and **b** are two edges. Since this parallelogram is the sum of two triangles of equal area, it follows that for Fig. 2.6 we have

$$2\ \mathrm{Area}(\Delta ABC) = |a \times b| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Note that this is valid only if A, B and C are labeled counter-clockwise; if this is not necessarily the case, we have to use the absolute value of $a_1 b_2 - a_2 b_1$. Since $a_1 = x_\mathrm{A} - x_\mathrm{C}$, $a_2 = y_\mathrm{A} - y_\mathrm{C}$, $b_1 = x_\mathrm{B} - x_\mathrm{C}$, $b_2 = y_\mathrm{B} - y_\mathrm{C}$, we can also write

$$2\,\mathrm{Area}(\Delta ABC) = (x_\mathrm{A} - x_\mathrm{C})(y_\mathrm{B} - y_\mathrm{C}) - (y_\mathrm{A} - y_\mathrm{C})(x_\mathrm{B} - x_\mathrm{C})$$

After working this out, we find that we can replace this with

$$2\,\mathrm{Area}(\Delta ABC) = (x_\mathrm{A} y_\mathrm{B} - y_\mathrm{A} x_\mathrm{B}) + (x_\mathrm{B} y_\mathrm{C} - y_\mathrm{B} x_\mathrm{C}) + (x_\mathrm{C} y_\mathrm{A} - y_\mathrm{C} x_\mathrm{A})$$

Although the latter expression seems hardly an improvement, it is useful to deduct a general form for computing the area of any polygon, convex or concave. For a polygon with vertices P_0, P_1, …, P_{n-1}, labeled counter-clockwise, we can derive:

$$2\ \mathrm{Area}(\mathrm{P}_0 \ldots \mathrm{P}_{n-1}) = \sum_{i=0}^{n-1} \left(x_i y_{i+1} - y_i x_{i+1}\right)$$

where (x_i, y_i) are the coordinates of P_i and P_n is the same vertex as P_0. As you can see, our last formula for the area of a triangle is a special case of this general one, in which the area of a polygon is expressed directly in terms of the coordinates of its vertices. A complete proof of this formula is beyond the scope of this book.

Java Code

As we have seen in Sect. 2.3, we use the method *area*2 to determine the orientation of three points A, B and C. Recall that the digit 2 in the name *area*2 indicates that we have to divide the return value by 2 to obtain the area of triangle ABC, that is, if A, B and C are counter-clockwise; otherwise, we have to take the absolute value |*area*2(A, B, C)/2|.

The same applies to the following method *area*2, whose return value divided by 2 gives the area, possibly preceded by a minus sign, of a polygon.

```
static float area2(Point2D[] pol) {
   int n = pol.length,
       j = n - 1;
   float a = 0;

   for (int i=0; i<n; i++) {
      // i == j + 1
      // (or j = n - 1 and i = 0)
      a += pol[j].x * pol[i].y - pol[j].y * pol[i].x;
      j = i;
   }
   return a;
}
```

Note that this second *area*2 method provides another means of deciding the orientation of a polygon vertex sequence: this orientation is counter-clockwise if *area*2 returns a positive value and clockwise if it returns a negative one. However, the method *ccw* discussed in Sect. 2.4 is faster, especially if *n* is large, because for most vertices it only performs the comparison

```
p[i].x <= p[k].x
```

while *area*2 performs some more time-consuming arithmetic operations for each vertex.

We could add this second method *area*2 to the class *Tools*2*D*, but you will not find it in our final version of this class because we will never use it in this book. Since we will use *Tools*2*D* several times, we prefer to omit superfluous methods for economic reasons.

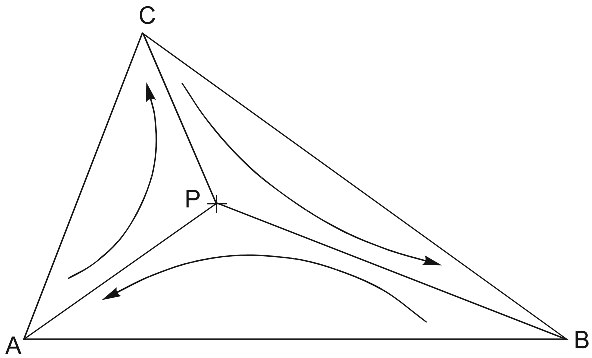
2.5 Point-in-Polygon Test

A Point Inside A Triangle

Determining the orientation of three points as we have just been discussing is useful in a test to see if a given point P lies within a triangle ABC. As Fig. 2.8 shows, this is the case if the orientation of the triangles ABP, BCP and CAP is the same as that of triangle ABC.

Let us assume that we know that the orientation of ABC is counter-clockwise. We can then call the following method to test if P lies within triangle ABC (or on an edge of it):

Fig. 2.8 Orientation used to test if P lies within triangle ABC




```
static boolean insideTriangle(Point2D a, Point2D b, Point2D c,
   Point2D p)  { // ABC is assumed to be counter-clockwise
   return
      Tools2D.area2(a, b, p) >= 0 &&
      Tools2D.area2(b, c, p) >= 0 &&
      Tools2D.area2(c, a, p) >= 0;
}
```

In this form, the method *insideTriangle* also returns the value *true* if P lies on an edge of the triangle ABC. If this is not desired, we should replace $> = 0$ with >0. The above form is the one we will need in later chapters, which is used to triangulate complicated polygonal faces of 3D objects.

Incidentally, with a floating-point value x, we might consider replacing a test of the form $x > = 0$ with $x > = -$ *eps* and $x > 0$ with x > *eps*, where *eps* is a small positive value, such as 10^{-6}. In this way a very small rounding-off error (less than *eps*) in the value of x will not affect the result of the test. This is not done here because the above method *insideTriangle* works well for our applications.

A Point Inside A Polygon

The notion of orientation is also useful when we need to determine whether a given point P lies within a polygon. It will then be convenient if a method is available which accepts both the polygon in question and point P as arguments, and returns *true* if P lies inside and *false* if it lies outside the polygon. If P lies on a polygon edge, we do not care: in that case the method may return *true* or *false*. The method we will discuss is based on the idea of drawing a horizontal half-line, which starts at P and extends only to the right. The number of intersections of this horizontal half-line with polygon edges is odd if P lies within the polygon and even if it lies outside it. Imagine that we move to the right, starting at point P. Then our state changes from *inside* to *outside* and vice versa each time we cross a polygon edge. The total

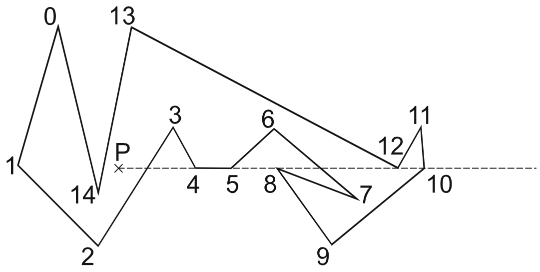


Fig. 2.9 Polygon and half-line starting at P

number of changes is therefore odd if P lies within the polygon and even if it lies outside the polygon. It is not necessary to visit all intersections strictly from left to right; the only thing we want to know is whether there are an odd or an even number of intersections on a horizontal line through P and to the right of P. However, we must be careful with special cases, shown in Fig. 2.9.

We simply ignore horizontal polygon edges, even if they have the same y-coordinate as P, as is the case with edge 4–5 in this example. If a vertex occurring as a 'local maximum or minimum' happens to have the same y-coordinate as P, as is the cases with the vertices 8 and 12 in this example, it is essential that this is either ignored or counted twice. We can realize this by using the edge from vertex i to vertex $i + 1$ only if

$$y_i \leq y_P < y_{i+1} \quad \text{or} \quad y_{i+1} \leq y_P < y_i$$

This implies that the lower endpoint of a non-horizontal edge is regarded as part of the segment, but the upper endpoint is not. For example, in Fig. 2.9, vertex 8 (with $y_8 = y_P$) is not counted at all because it is the upper endpoint of both edge 7–8 and edge 8–9. By contrast, vertex 12 (with $y_{12} = y_P$) is the lower endpoint of the edges 11–12 and 12–13 and thus counted twice. Therefore, in this example, we count the intersections of the half-line through P with the seven edges 2–3, 3–4, 5–6, 6–7, 10–11, 11–12, 12–13 and with no others.

Since we are considering only a half-line, we must impose another restriction on the set of edges satisfying the above test, selecting only those whose point of intersection with the half-line lies to the right of P. One way of doing this is by using the method *area*2 to determine the orientation of a sequence of three points. For example, this orientation is counter-clockwise for the triangle 2–3-P in Fig. 2.9, implying that P lies to the left of edge 2–3. It is also counter-clockwise for the triangle 7–6-P. In both cases, the lower endpoint of an edge, its upper endpoint and point P, in that order, are counter-clockwise. The following method is based on these principles:

```
static boolean insidePolygon(Point2D p, Point2D[] pol)
{  int n = pol.length, j = n - 1;
   boolean b = false;
   float x = p.x, y = p.y;
```

```
   for (int i=0; i<n; i++)
   {  if (pol[j].y <= y && y < pol[i].y &&
          Tools2D.area2(pol[j], pol[i], p) > 0 ||
          pol[i].y <= y && y < pol[j].y &&
          Tools2D.area2(pol[i], pol[j], p) > 0) b = !b;
      j = i;
   }
   return b;
}
```

This static method, like some others in this chapter, could be added to the class *Tools2D* (see Sect. 2.3).

The Contains *Method of* Polygon *Class*

There is a standard class *Polygon*, which has a member named *contains* to perform about the same task as the above method *insidePolygon*. However, it is based on integer coordinates. For example, to test if a point P(*xP*, *yP*) lies within the triangle with vertices A(20, 15), B(100, 30) and C(80, 150), we can use the following fragment:

```
int[] x = {20, 100, 80}, y = {15, 30, 150}; // A, B, C
Polygon p = new Polygon(x, y, 3);
if (p.contains(xP, yP)) ... // P lies within triangle ABC.
```

If P lies exactly on a polygon edge, the value returned by this method *contains* of the class *Polygon* can be *true* or *false*.

2.6 Triangulation of Polygons

In many graphics applications, such as those to be discussed in Chap. 6, it is desirable to divide a polygon into triangles. This triangulation problem can be solved in many ways. We will discuss an algorithm that attempts to produce triangles without very small angles. In other words, we want these triangles to approximate equilateral ones. More precisely, the smallest of the angles occurring in all these triangles should be as large as possible. The work will be done by the method *triangulate* of a class *Polygon2D*. A polygon, given in the form of an array of *Point2D* elements (see Sect. 1.4), and containing the polygon vertices in counter-clockwise order, is stored in this class. The resulting triangles will be stored in an array, each element of which is of type *Tria*, defined as

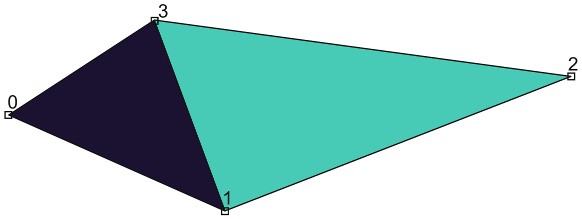


Fig. 2.10 Triangulation of a quadrangle

```
class Tria {
   int iA, iB, iC;
   Tria(int i, int j, int k){iA = i; iB = j; iC = k;}
}
```

and containing the three vertex numbers of a triangle. So we will perform triangulation as follows:

```
Point2D[] p = new Point2D[n];
... // Store the polygon in array p
Polygon2D polygon = new Polygon2D(p);
Tria[] t = polygon.triangulate();
```

Figure 2.10 shows a polygon with four vertices, known as a *quadrangle*, with vertex numbers 0, 1, 2, 3. In this case, the array elements $p[j]$ contain the *Point2D* object for the coordinates of vertex j ($j = 0, 1, 2, 3$). After creating the object *polygon* and using the *triangulate* method to produce array t, as shown in the above program fragment, we have

$$t[0].iA = 0,\ t[0].iB = 1, t[0].iC = 3$$
$$t[1].iA = 3,\ t[1].iB = 1, t[2].iC = 2$$

Note that vertices are visited in counter-clockwise order, both for the quadrangle and for the two triangles. Another interesting point is the choice of diagonal 1–3. If, instead, diagonal 0–2 had been chosen, much smaller angles would have occurred at the vertices 0 and 2. Finally, we see that the number of triangles is 2 less than n, the number of polygon vertices. This is a general rule: a polygon with n vertices is divided into $n - 2$ triangles.

In the first part of the algorithm we do not yet worry about the choice of the diagonals, other than the obvious requirement that they must completely lie inside the polygon. It works as follows. Traversing the vertices of the polygon in counter-clockwise order, for every three successive vertices P, Q and R of which Q is a convex vertex (interior angle less than 180°), we cut the triangle PQR off the polygon if this triangle does not contain any of the other polygon vertices. For example, starting with polygon ABCDE in Fig. 2.11, we cannot cut triangle ABC, because ABC contains vertex D. Nor is triangle CDE a good candidate, because D

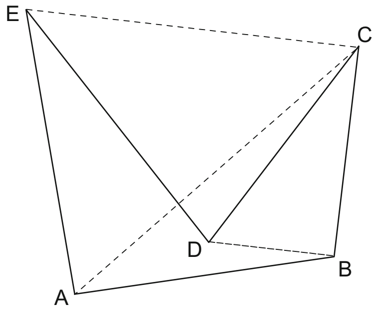


Fig. 2.11 Cutting off a triangle

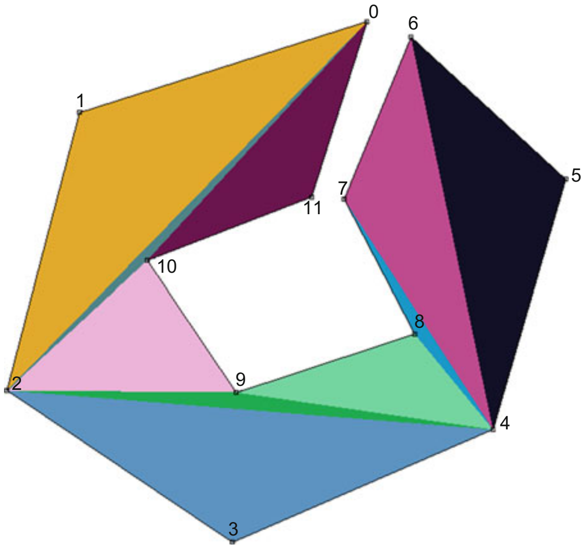


Fig. 2.12 Too small angles in some triangles

is not a convex vertex. There are no such problems with triangle BCD, so that we will cut this off the polygon, reducing the polygon ABCDE to the simpler one ABDE.

Proceeding in the same way with the polygon ABDE and so on, we will succeed in dividing the polygon into triangles, but, in general, this way of triangulating may produce awkward results because of some very small angles, as demonstrated in Fig. 2.12.

Instead, we prefer triangles of which the smallest angle is as large as possible. Such a division of polygons (or, in general, a set of points) is known as *Delaunay triangulation*, named after Boris Nikolaevich Delaunay (1890–1980). Figure 2.13 demonstrates this.

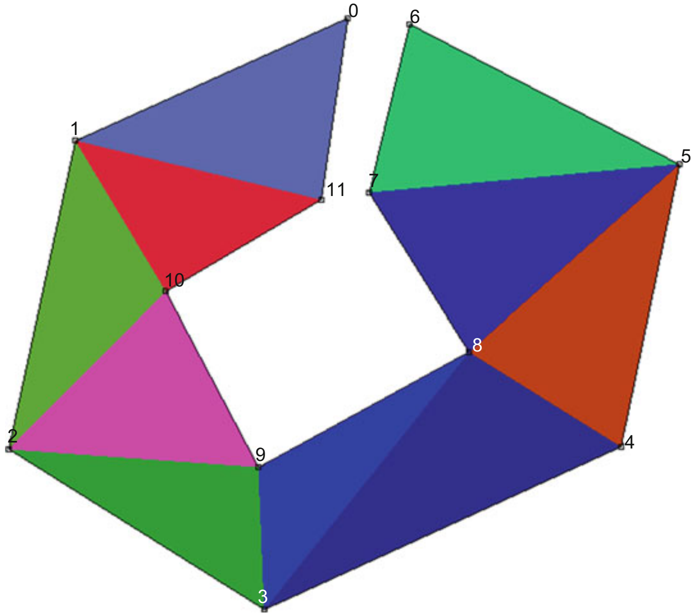


Fig. 2.13 Delaunay triangulation

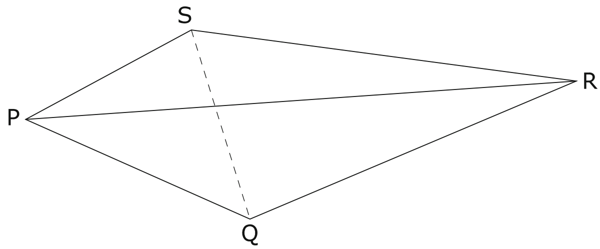


Fig. 2.14 Diagonal flipping: replacing PR with QS

We will see later that triangulation is used in 3D graphics to approximate curved surfaces. Such approximations are best done with Delaunay triangulation. Surprisingly, this can be achieved in an elegant way, known as *diagonal flipping*. Starting with a set of triangles produced in a naïve way, we look for a quadrangle PQRS, with diagonal PR as a common edge of the two adjacent triangles PQR and RSP, and we look if there are better triangles with diagonal QS instead of PR. This is clearly the case in Fig. 2.14: we prefer the triangles PQS and SQR to those just mentioned. In general, replacing PR with QS is desirable if (and only if) the sum of the angles Q and S is greater than 180°. A proof of this is beyond the scope of this book.

After having replaced two triangles with two better ones that form the same quadrangle, we repeat the search process, searching all triangles again. After all, a triangle created by diagonal flipping may later successfully be combined with another adjacent triangle. This is implemented in the class *Polygon2D*, listed below.

```
// Polygon2D: A class to triangulate a polygon.
import java.util.Vector;

public class Polygon2D {
   Point2D[] vertices;
   int[] nrs;
   int n;

   Polygon2D(Point2D[] p) {
      vertices = p;
      n = p.length;
      nrs = new int[n];
      for (int i=0; i<n; i++)
         nrs[i] = i;
   }

   Polygon2D(Point2D[] p, int[] index) { // Used for 3D applications
      vertices = p;
         n = index.length;
         nrs = index;
   }

   float angle(Point2D a, Point2D b, Point2D c){
     // Return angle ABC, using dot product u . v = |u||v|cos angle
     double xBA = a.x - b.x, yBA = a.y - b.y,
            xBC = c.x - b.x, yBC = c.y - b.y,
            dotproduct = xBA * xBC + yBA * yBC,
            lenBA = Math.sqrt(xBA * xBA + yBA * yBA),
            lenBC = Math.sqrt(xBC * xBC + yBC * yBC),
            cosB = dotproduct / (lenBA * lenBC);
     return (float)Math.acos(cosB);
   }

   boolean flippingDesirable(int iP, int iQ, int iR, int iS) {
      // Currently, diagonal PR divides quadrangle PQRS into
      // two triangles. Is the alternative diagonal QS a
      // better choice?
      Point2D vP = vertices[iP], vQ = vertices[iQ],
              vR = vertices[iR], vS = vertices[iS];
```

```
      // Compute the angles at the opposite vertices Q and S.
      // Flipping is desirable if (angle Q) + (angle S) > pi
      return angle(vP, vQ, vR) + angle(vR, vS, vP) > Math.PI;
   }

   boolean anyFlipping(Tria[] trias) {
      if (trias.length < 2)
         return false;
      for (int i=0; i<trias.length; i++){
         int[] t = {trias[i].iA, trias[i].iB, trias[i].iC};
         for (int j=i+1; j<trias.length; j++){
            int[] u = {trias[j].iA, trias[j].iB, trias[j].iC};
            // Look for a common edge of triangles t and u
            for (int h=0; h<3; h++){
               for (int k=0; k<3; k++){
                  if (t[h] == u[k] &&
                    t[(h+1)%3] == u[(k+2)%3]){
                     int iP = t[(h+1)%3], iQ = t[(h+2)%3],
                         iR = t[h], iS = u[(k+1)%3];
                     if (flippingDesirable(iP, iQ, iR, iS)){
                        trias[i] = new Tria(iP, iQ, iS);
                        trias[j] = new Tria(iS, iQ, iR);
                        return true;
                     }
                  }
               }
            }
         }
      }
      return false;
   }

   Tria[] triangulate(){
      Tria[] tr = new Tria[n - 2];
      int[] next = new int[n];
      for (int i = 0; i < n; i++)
         next[i] = (i + 1) % n;

      for (int k = 0; k < n - 2; k++) {
         // Find a suitable triangle, consisting of two edges
         // and an internal diagonal:
         Point2D a, b, c;
         boolean triaFound = false;
         int iA = 0;
```

```
         int count = 0;
         while (!triaFound && ++count < n) {
            int iB = next[iA], iC = next[iB];
            int nA = Math.abs(nrs[iA]),
                nB = Math.abs(nrs[iB]),
                nC = Math.abs(nrs[iC]);
            a = vertices[nA];
            b = vertices[nB];
            c = vertices[nC];
            if (Tools2D.area2(a, b, c) >= 0) {
               // Edges AB and BC; diagonal AC.
               // Test to see if no other polygon vertex
               // lies within triangle ABC:
               int j = next[iC];
               int nj = Math.abs(nrs[j]);
               while (j != iA &&
                    (nj == nA || nj == nB || nj == nC) ||
                     !Tools2D.insideTriangle(a, b, c,
                        vertices[nj])){
                  j = next[j];
                  nj = Math.abs(nrs[j]);
               }
               if (j == iA) {
                  // Triangle ABC contains no other vertex:
                  tr[k] = new Tria(nA, nB, nC);
                  next[iA] = iC;
                  triaFound = true;
               }
            }
            iA = next[iA];
         }

         if (count == n)
         {  System.out.println("Not a simple polygon" +
              " or vertex sequence not counter-clockwise.");
            System.exit(1);
         }
      }

      while (anyFlipping(tr))
         ; // Keep flipping diagonals as long as this is
           // desirable
      return tr;
   }
}
```

You may have noticed that this class contains two *Polygon2D* constructors. In this chapter, we will use only the first. When dealing with 3D objects in Chap. 6, we will use the second, which is useful if we want to use only one array of *Point2D* objects to store the vertices of several polygons. Then the second argument, *index*, of this constructor, indicates which of all those vertices comprise the polygon that is to be triangulated. Actually, the first constructor, with only the array *p* as an argument, builds such an array, internally named *nrs*. Initially, for a polygon of *n* vertices, it is assumed that this polygon has the vertex numbers 0, 1, . . ., $n - 1$, which are stored in the array *nrs*. So both now and in later chapters, the polygon has the following vertex numbers:

```
vertices[nrs[0]], vertices[nrs[1]],..., vertices[nrs[n - 1]]
```

We also use the array *next*, so we can easily deal with the vertices in counter-clockwise order. Initially,

$$next[i] = i + 1 \,\text{for}\, i = 0, 1, \ldots, n - 2$$
$$next[i - 1] = 0$$

This is achieved by means of the following fragment:

```
for (int i = 0; i < n; i++)
   next[i] = (i + 1) % n;
```

This use of the modulo operator % for cyclic traversing the vertices of polygons and triangles is often used in this program, so it is essential to understand it. The introduction of this array *next* enables us to deal with reduced polygons after triangles have been cut off. You can see this happening in the fragment.

```
if (j == iA) { // Triangle ABC contains no other vertex:
   tr[k] = new Tria(nA, nB, nC);
    next[iA] = iC;
   triaFound = true;
}
```

For example if the polygon has 5 vertices, numbered 0, 1, 2, 3, 4, and triangle 2-3-4 is cut off the polygon, vertex 4 replaces vertex 3 as the successor of vertex 2, so we make $next[2] = 4$.

To see this class *Polygon2D* and the *triangulate* method in action, we will use a demonstration program, *PolyTria.java*. It allows the user to define a polygon, in the same way as we did with program *DefPoly.java* in Sect. 1.4, but this time this polygon will be divided into triangles, which appear in different colors.

This program, *PolyTria.java*, uses the above classes *Tria* and *Tools2D*, as well as the class *CvDefPoly*, which occurs in program *DefPoly.java* of Sect. 1.4. In our subclass, *CvPolyTria*, we apply the method *ccw*, discussed in Sect. 2.4, to the given polygon to examine the orientation of its vertex sequence. If this happens to be clockwise, we put the vertices in reverse order in the array *p*, so that the vertex sequence will be counter-clockwise in this array, which we can then safely pass on to the *Polygon2D* constructor:

```
// PolyTria.java: Drawing a polygon and dividing it into triangles.
// Uses: CvDefPoly, Point2D (Section 1.4),
//       Tools2D (Section 2.3), Tria, Polygon2D (discussed above).

import java.awt.*;
import java.awt.event.*;

public class PolyTria extends Frame {
   public static void main(String[] args) {new PolyTria();}

   PolyTria() {
      super("Define polygon vertices by clicking");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {
            System.exit(0);}
      });
      setSize(1500, 900);
      add("Center", new CvPolyTria());
      setCursor(Cursor.getPredefinedCursor
      (Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvPolyTria extends CvDefPoly { // see Section 1.4
   public void paint(Graphics g) {
      int n = v.size(); // v is defined in superclass CvDefPoly
      if (n >= 3 && ready) {
         Point2D[] p = new Point2D[n];
         for (int i = 0; i < n; i++)
            p[i] = (Point2D) v.elementAt(i);
         // If not counter-clockwise, reverse the order:
         if (!ccw(p))
            for (int i = 0; i < n; i++)
               p[i] = (Point2D) v.elementAt(n - i - 1);
```

```
         Polygon2D polygon = new Polygon2D(p);
         Tria[] t = polygon.triangulate();
         initgr();
         if (t != null) {
            for (int j = 0; j < t.length; j++) {
               g.setColor(new Color(rand(), rand(), rand()));
               int iA = t[j].iA, iB = t[j].iB, iC = t[j].iC;
               int[] x = new int[3], y = new int[3];
               x[0] = iX(p[iA].x); y[0] = iY(p[iA].y);
               x[1] = iX(p[iB].x); y[1] = iY(p[iB].y);
               x[2] = iX(p[iC].x); y[2] = iY(p[iC].y);
               g.fillPolygon(x, y, 3);
            }
         }
      }
      g.setColor(Color.black);
      super.paint(g);
   }

   int rand() {return (int) (Math.random() * 256);}

   static boolean ccw(Point2D[] p) {
      int n = p.length, k = 0;
      for (int i = 1; i < n; i++)
         if (p[i].x <= p[k].x &&
            (p[i].x < p[k].x || p[i].y < p[k].y))
            k = i;
      // p[k] is a convex vertex.
      int prev = k - 1, next = k + 1;
      if (prev == -1) prev = n - 1;
      if (next == n) next = 0;
      return Tools2D.area2(p[prev], p[k], p[next]) > 0;
   }
}
```

The canvas class *CvPolyTria* is a subclass of *CvDefPoly* so that the construction of a polygon with vertices specified by the user is done in the same way as in Sect. 1.4. In this subclass, we call the method *triangulate* to construct the array *tr* of triangles. These are then displayed in colors generated with a random number generator, so that we can clearly distinguish them, as shown in Fig. 2.15.

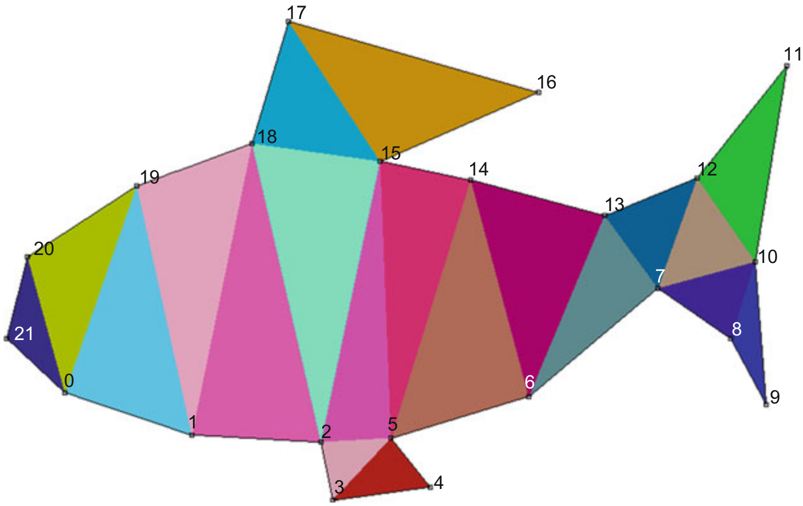


Fig. 2.15 Triangulation of a polygon

2.7 Point-on-Line Test

Testing whether a point P lies on a given line is very simple if this line is given as an equation, say,

$$ax + by = h \tag{2.3}$$

Then all we need to do is to test whether the coordinates of P satisfy this equation. Due to the discrete nature of the screen (as discussed in Chap. 1), such a test may fail if P is not exactly one of the pixels on the line. It may therefore be wise to be a little tolerant, so that we may write.

```
if (Math.abs(a * p.x + b * p.y - h) < eps) // P on the line
```

where *eps* is a small positive value, such as 10^{-5}.

If the line is not given by an equation but by two points A and B on it, we can use the above test after deriving an equation for the line, writing

$$\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0$$

which gives the following coefficients for Eq. (2.3):

$$a = y_A - y_B$$
$$b = x_B - x_A$$
$$h = x_B y_A - x_A y_B$$

Instead, we can benefit from the *area*2 method. After all, if and only if P lies on the line through A and B, the triangle ABP is degenerated and has a zero area. We can therefore write.

```
if (Math.abs(Tools2D.area2(a, b, p)) < eps) // P on line AB
```

A Point on a Line Segment

Given three points A, B and P, we may want to determine whether P lies on the closed line segment AB. The adjective *closed* here means that we include the endpoints A and B, so that the question is to be answered affirmatively if P lies between A and B or coincides with one of these points. We assume that A and B are different points, i.e., $x_A \neq x_B$ or $y_A \neq y_B$. If $x_A \neq x_B$ we test whether x_P lies between x_A and x_B; if not, we test whether y_P lies between y_A and y_B, where in both cases the word *between* includes the points A and B themselves. This test is sufficient if P lies on the infinite line AB. Otherwise, we also have to perform the above test, which is done in the following method:

```
static boolean onSegment(Point2D a, Point2D b, Point2D p) {
   double eps = 1e-2;
   return
   (a.x != b.x &&
      (a.x <= p.x && p.x <= b.x || b.x <= p.x && p.x <= a.x)
   || a.x == b.x &&
   (a.y <= p.y && p.y <= b.y || b.y <= p.y && p.y <= a.y))
   && Math.abs(Tools2D.area2(a, b, p)) < eps;
}
```

The expression following *return* relies on the and-operator && having higher precedence than the or-operator ||. Since both operators && and | | evaluate the second operand only if it is necessary, this test is more efficient than it looks. For example, if $x_A \neq x_B$ the test on the line of the form $(a.y <= \ldots)$ is not evaluated at all. The positive constant $eps = 1e\text{–}2$ $(= 10^{-2})$ in the above code may be replaced with a smaller or larger one, depending on the application needs.

Instead of testing if P lies on the segment AB, we may want to apply a similar test to the projection P′ of P on AB, as Fig. 2.16 shows.We can solve this problem by computing the dot product of the vectors **AB** and **AP**. This dot product $\mathbf{AB} \cdot \mathbf{AP}$ is equal to 0 if P′ ° A (P′ coincides with A) and it is equal to $\mathbf{AB} \cdot \mathbf{AB} = \mathrm{AB}^2$ if

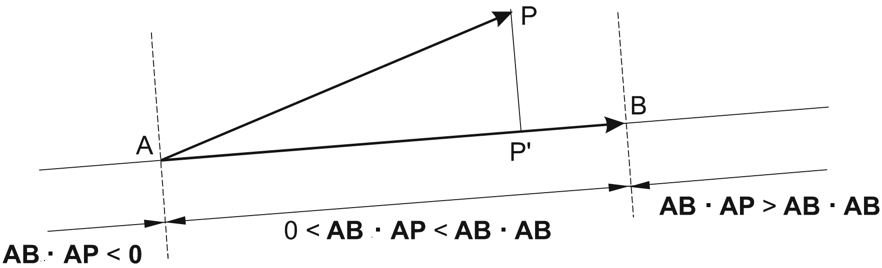


Fig. 2.16 Projection P′ of P on line AB between A and B

P′ ° B. For any value of this dot product between these two values, P lies between A and B. We can write this as follows in a program, where *len2* corresponds to **AB** · **AB,** *inprod* corresponds to **AB** · **AP,** and *eps* is some very small positive value:

```
// Does P′ (P projected on AB) lie on the closed segment AB?
static boolean projOnSegment(Point2D a, Point2D b, Point2D p) {
   double eps = 1e-2,
      ux = b.x - a.x, uy = b.y - a.y,
      len2 = ux * ux + uy * uy,
      inprod = ux * (p.x - a.x) + uy * (p.y - a.y);
   return inprod > -eps && inprod < len2 + eps;
}
```

To determine whether P′ lies on the *open* segment AB (not including A and B), we replace the return statement with

```
return inprod > eps && inprod < len2 - eps;
```

2.8 Projection of a Point on a Line

Suppose that again a line l and a point P (not on l) are given and that we want to compute the projection P′ of P on l (see Fig. 2.16). This point P′ has three interesting properties:

1. P′ is the point on l that is closest to P.
2. The length of PP′ is the distance between P and l (see also Sect. 2.9).
3. PP′ and l are perpendicular.

We discuss two solutions: one for a line l given by two points A and B, and the other for l given as the equation $\mathbf{x} \cdot \mathbf{n} = h$.

With given points A and B on line l, the situation is as shown in Fig. 2.16. Recall that in Sect. 2.7 we discussed the method *projOnSegment* to test if the projection P′ of P on the line through A and B lies between A and B. In that method, we did not actually compute the position of P′. We will now do this (see Fig. 2.16), first by introducing the vector **u** of length 1 and direction **AB**:

$$\mathbf{u} = \frac{1}{|\mathbf{AB}|}\ \mathbf{AB}$$

Then the length of the projection AP′ of AP is equal to

$$\lambda = \mathbf{AP} \cdot \mathbf{u}$$

which we use to compute

$$\mathbf{AP}' = \lambda\mathbf{u}$$

Doing this straightforwardly would require a square-root operation in the computation of the distance between A and B, used in the computation of **u**. Fortunately, we can avoid this by rewriting the last equation, using the two preceding ones:

$$\mathbf{AP}' = (\mathbf{AP} \cdot \mathbf{u})\ \mathbf{u} = \left(\mathbf{AP} \cdot \frac{1}{|\mathbf{AB}|}\mathbf{AB}\right)\ \frac{1}{|\mathbf{AB}|}\mathbf{AB} = \frac{1}{|\mathbf{AB}|^2}(\mathbf{AP} \cdot \mathbf{AB})\ \mathbf{AB}$$

The advantage of the last form is that the square of the segment length AB is easier to compute than that length itself. The following method, which returns the projection P′ of P on AB, demonstrates this:

```
// Compute P' (P projected on AB):
static Point2D projection(Point2D a, Point2D b, Point2D p) {
   float vx = b.x - a.x, vy = b.y - a.y, len2 = vx * vx + vy * vy,
     inprod = vx * (p.x - a.x) + vy * (p.y - a.y);
   return new Point2D(a.x + inprod * vx/len2,
                      a.y + inprod * vy/len2);
}
```

Let us now consider a line l given by its equation, which we again write as

$$\mathbf{x} \cdot\ \mathbf{n} = h$$

where

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\sqrt{a^2 + b^2} = 1$$

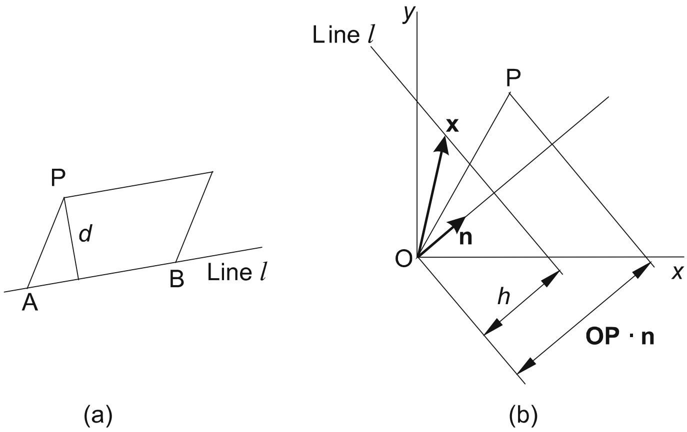


Fig. 2.17 Distance between point P and line l

Using the 'signed distance'

$$d = \mathbf{OP} \cdot \mathbf{n} - h$$

as illustrated by Fig. 2.17 of the next section, we can write the following vector equation to compute the desired projection P′ of P on l:

$$\mathbf{OP}' = \mathbf{OP} - d\mathbf{n} = \begin{pmatrix} x_P \\ y_P \end{pmatrix} - d\begin{pmatrix} a \\ b \end{pmatrix}$$

This should make the following method clear:

```
// Compute P', the projection of P on line l given as
// ax + by = h, where a * a + b * b = 1
static Point2D projection(float a, float b, float h, Point2D p) {
   float d = p.x * a + p.y * b - h;
   return new Point2D(p.x - d * a, p.y - d * b);
}
```

2.9 Distance Between a Point and a Line

We can find the distance between a point P and a line l in different ways, depending on the way the line is specified. If two points A and B of the line are given, we can find the distance between a point P and the (infinite) line l through A and B by using the method *Tools2D.area2* defined in Sect. 2.3:

$$\text{distance between P and line AB} = d = \frac{|area2(\mathrm{A}, -\mathrm{B}, -\mathrm{P})|}{|\mathrm{AB}|}$$

This follows from the fact that the absolute value of *area*2(A, B, P) denotes the area of the parallelogram formed by A, B and P, as shown in Fig. 2.17a. This area is also equal to the product of the parallelogram's base AB and its height d. We can therefore compute d in the above way.

If the line l is given as an equation, we assume this to be in the form

$$ax + by = h$$

where

$$\sqrt{a^2 + b^2} = 1$$

If the latter condition is not satisfied, we only have to divide a, b and h by the above square root. We can then write the above equation of line l as the dot product

$$\mathbf{x} \cdot \mathbf{n} = h$$

where

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The normal vector $\mathbf{n}$ is perpendicular to line l and has length 1. For any vector $\mathbf{x}$, starting at O, the dot product $\mathbf{x} \cdot \mathbf{n}$ is the projection of vector $\mathbf{x}$ on $\mathbf{n}$. This also applies if the endpoint of $\mathbf{x}$ lies on line l, as shown in Fig. 2.17b; in this case we have $\mathbf{x} \cdot \mathbf{n} = h$. We find the desired distance between point P and line l by projecting $\mathbf{OP}$ also on $\mathbf{n}$ and computing the difference of the two projections:

$$\text{Distance between P and line } l = |\mathbf{OP} \cdot \mathbf{n} - h| = |ax_{\mathrm{P}} + by_{\mathrm{P}} - h|$$

Although Fig. 2.17b applies to the case $h > 0$, this equation is also valid if h is negative or zero, or if O lies between line l and the line through P parallel to l. Both $\mathbf{OP} \cdot \mathbf{n}$ and h are scale factors for the same vector $\mathbf{n}$. The absolute value of the algebraic difference of these two scale factors is the desired distance between P and l.

Exercises

2.1. Write a program that draws a square ABCD. The points A and B are arbitrarily specified by the user by clicking the mouse button. The orientation of the points A, B, C and D should be counter-clockwise.

2.2. Write a program that, for four points A, B, C and P,

- draws a triangle formed by ABC and a small cross showing the position of P; and

- displays a line of text indicating which of the following three cases applies: P lies (a) inside ABC, (b) outside ABC, or (c) on an edge of ABC.

  The user will specify the four points by clicking.

2.3. The same as Exercise 2.2, but, instead of displaying a line of text, the program computes the distances of P to the (infinite) lines AB, BC and CA, and draws the shortest possible line that connects P with the nearest of those three lines.
2.4. Write a program that computes the intersection point of two (infinite) lines AB and CD. The user will specify the points A, B, C and D by clicking. Draw a small circle around the intersection point. If the two lines AB and CD do not have a unique intersection point (because they are parallel or coinciding), display a line of text indicating this.
2.5. Write a program that constructs the bisector of the angle ABC (which divides the angle at B into two equal angles). After the user has specified the points A, B and C by clicking, the program should compute the intersection point D of this bisector with the opposite side AC, and draw both triangle ABC and bisector BD.
2.6. Write a program that draws the *circumscribed circle* (also known as the *circumcircle*) of a given triangle ABC; this circle passes through the points A, B and C. These points will be specified by the user by clicking the mouse button. Remember, the three perpendicular bisectors of the three edges of a triangle all pass through one point, the *circumcenter*, which is the center of the circumscribed circle.
2.7. Write a program that, for three given points P, Q and R specified by the user, draws a circular arc, starting at P, passing through Q and ending at R (see also Exercise 2.1).
2.8. Construct a fillet to replace a sharp corner with a rounded one, as illustrated by the solid lines and the arc in Fig. 2.14. The four points A, B, C and D are specified by the user by clicking the mouse. Point D may or may not lie on AB; if it does not, it is projected onto AB, giving D′, as Fig. 2.18 illustrates. The arc starts at point D′.

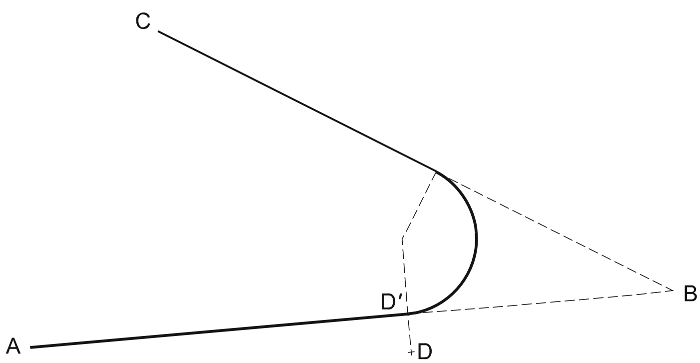


Fig. 2.18 Fillet

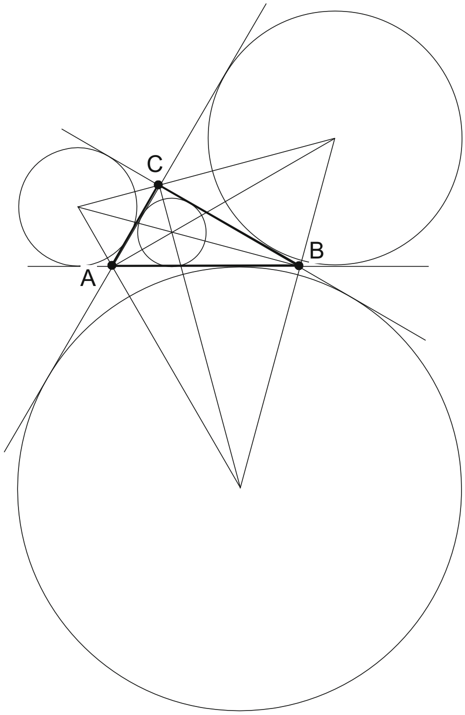


Fig. 2.19 Incircle and excircles of triangle ABC

2.9. Construct the *inscribed circle* (or *incircle*) of a given triangle ABC. The center of this circle lies on the point of intersection of the (internal) bisectors of the three angles A, B and C. Draw also the three *excircles*, which, like the incircle, are tangent to the sides of the triangle, as shown in Fig. 2.19. The centers of the excircles lie on the points of intersection of the external bisectors of the angles A, B and C.

2.10. Write a program that draws a tree of Pythagoras as shown in Fig. 2.20. Two vertices A and B, the basis of the tree, are specified by the user by pressing a mouse button. Then both a square ABCD and an isosceles, right-angled triangle DCE, with right angle E, is constructed. The orientation of both

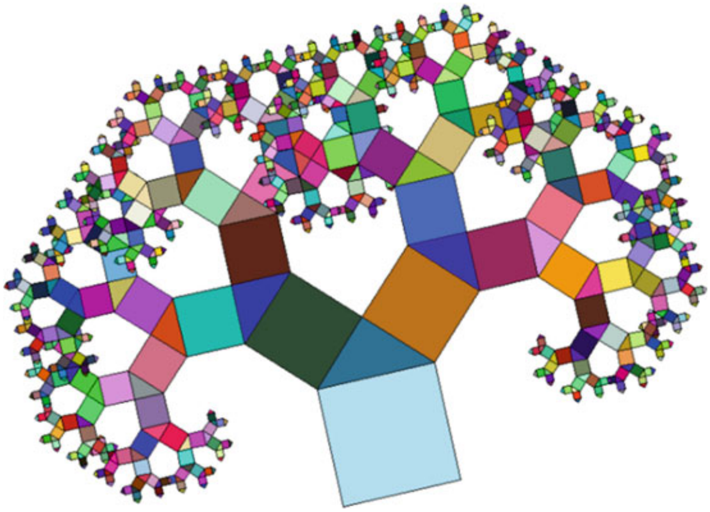


Fig. 2.20 Tree of Pythagoras

ABCD and DCE is counter-clockwise. Finally, the points D and E form the basis of another tree of Pythagoras, and so do the points E and C. Use a recursive method, which does nothing at all if the two supplied basis points are closer together than some limit.

Chapter 3
Geometrical Transformations

To understand perspective projection, to be discussed in Chap. 5, we need to be familiar with 3D rotations. These and other transformations will be discussed in this chapter. They are closely related to matrix multiplication, which is the subject we start with.

3.1 Matrix Multiplication

A *matrix* (plural *matrices*) is a rectangular array of numbers enclosed in brackets (or parentheses). For example,

$$\begin{bmatrix} 2 & 0 & 0.1 & 3 \\ 1 & 4 & 2 & 10 \end{bmatrix}$$

is a 2×4 matrix: it consists of two rows and four columns. If a matrix consists of only one row, we call it a *row matrix* or *row vector*. In the same way, we use the term *column matrix* or *column vector* for a matrix that has only one column.

If A and B are matrices and the number of columns of A is equal to the number of rows of B, we can compute the *matrix product* AB. This product is another matrix, which has as many rows as A and as many columns as B. We will discuss this in detail for a particular case with regard to the dimensions of A and B: we will use a 2×3 matrix A and a 3×4 matrix B. Then the product $C = AB$ exists and is a 2×4 matrix. It will be clear that the matrix product AB can be computed for A and B of other dimensions in a similar way, provided the number of columns of A is equal to the number of rows of B.

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2_3

Writing

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and similar expressions for both the matrix B and the product matrix $C = AB$, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{23} \end{bmatrix}$$

Each element c_{ij} (found in row i and column j of the product matrix C) is equal to the dot product of the ith row of A and the jth column of B. For example, to find c_{23}, we need the numbers a_{21}, a_{22} and a_{23} in the second row of A and b_{13}, b_{23} and b_{33} in the third column of B. Using these two sequences as vectors, we can compute their dot product, finding

$$c_{23} = (a_{21}, a_{22}, a_{23}) \cdot (b_{13}, b_{23}, b_{33}) = (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}).$$

In general, the elements of the above matrix C are computed as follows:

$$c_{ij} = (a_{i1}, a_{i2}, a_{i3}) \cdot \left(b_{1j}, b_{2j}, b_{3j}\right) = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}$$

3.2 Linear Transformations

A transformation T is a mapping

$$\mathbf{v} \rightarrow T\mathbf{v} = \mathbf{v}'$$

such that each vector $\mathbf{v}$ (in the vector space we are dealing with) is assigned its unique image $\mathbf{v}'$. Let us begin with the xy-plane and associate with each vector $\mathbf{v}$ the point P, such that

$$\mathbf{v} = \mathbf{OP}$$

Then the transformation T is also a mapping

$$\mathrm{P} \rightarrow \mathrm{P}'$$

for each point P in the xy-plane, where $\mathbf{OP}' = \mathbf{v}'$.

A transformation is said to be *linear* if the following is true for any two vectors **v** and **w** and for any real number λ:

$$T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$$
$$T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$$

By using $\lambda = 0$ in the last equation, we find that, for any linear transformation, we have

$$T(\mathbf{0}) = \mathbf{0}$$

We can write any linear transformation as a matrix multiplication. For example, consider the following linear transformation:

$$\begin{cases} x' = 2x \\ y' = x + y \end{cases}$$

We can write this as the matrix product

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.1}$$

or as the following:

$$[x' \quad y'] = [x \quad y] \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \tag{3.2}$$

The above notation (3.1) is normally used in standard mathematics textbooks; in computer graphics and other applications in which transformations are combined, the notation (3.2) is also popular because it avoids a source of mistakes, as we will see in a moment. We will therefore adopt this notation, using row vectors.

It is interesting to note that, in (3.2), the rows of the 2×2 transformation matrix are the images of the unit vectors (1, 0) and (0, 1), respectively, while these images are the columns in (3.1). You can easily verify this by substituting [1 0] and [0 1] for [x y] in (3.2), as the bold matrix elements below illustrate:

$$[\mathbf{2} \quad \mathbf{1}] = [1 \quad 0] \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ 0 & 1 \end{bmatrix}$$
$$[\mathbf{0} \quad \mathbf{1}] = [0 \quad 1] \begin{bmatrix} 2 & 1 \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

This principle also applies to other linear transformations. It provides us with a convenient way of finding the transformation matrices.

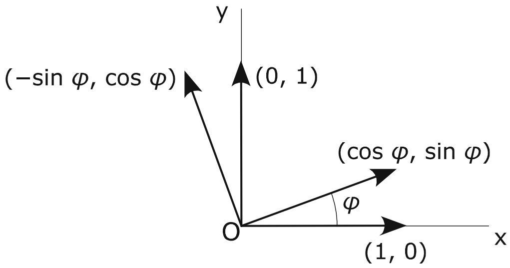


Fig. 3.1 Rotation of unit vectors

Rotation

To rotate all points in the *xy*-plane about O through the angle φ, we can now easily write the transformation matrix, using the rule we have just been discussing. We simply find the images of the unit vectors (1, 0) and (0, 1). As we know from elementary trigonometry, rotating the points P(1, 0) and Q(0, 1) about O through the angle φ gives P′(cos φ, sin φ) and Q′(−sin φ, cos φ). It follows that (cos φ, sin φ) and (−sin φ, cos φ) are the desired images of the unit vectors (1, 0) and (0, 1), as Fig. 3.1 illustrates.

Then all we need to do is to write these two images as the rows of our rotation matrix:

$$[x' \quad y'] = [x \quad y]\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \tag{3.3}$$

A Programming Example

To see rotation in action, let us rotate an arrow about the origin O. Before this rotation, the arrow is vertical, points upward and can be found to the right of O. We will rotate this angle through 120° about the origin O, which is the center of the canvas. Figure 3.2 shows the coordinate axes (intersecting in O) and the arrow before and after the rotation.

If we change the dimensions of the window, the origin remains in the center and the sizes of the arrows and of the circle on which they lie change accordingly, while this circle remains a circle.

Recall that we have also placed the origin in the center of the canvas in the *Isotrop.java* program in Sect. 1.3, when dealing with the isotropic mapping mode. The following program also uses this mapping mode and contains the same methods *iX* and *iY* for the conversion from logical to device coordinates:

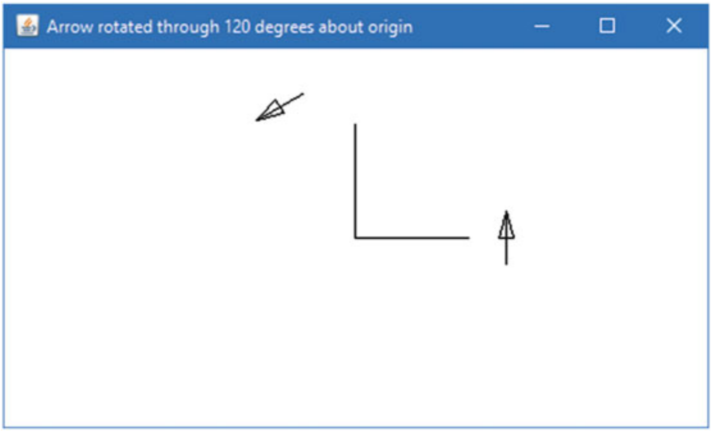


Fig. 3.2 Arrow before and after rotation through 120° about the origin

```
// Arrow.java: Arrow rotated through 120 degrees about the logical
//             origin O, which is the center of the canvas.
import java.awt.*;
import java.awt.event.*;

public class Arrow extends Frame {
   public static void main(String[] args) {new Arrow();}

   Arrow() {
      super("Arrow rotated through 120 degrees about origin");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(500, 300);
      add("Center", new CvArrow());
      setVisible(true);
    }
}

class CvArrow extends Canvas {
   int centerX, centerY, currentX, currentY;
   float pixelSize, rWidth = 100.0F, rHeight = 100.0F;

   void initgr() {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
   }
```

```
   int iX(float x) {return Math.round(centerX + x / pixelSize);}
   int iY(float y) {return Math.round(centerY - y / pixelSize);}

   void moveTo(float x, float y) {currentX = iX(x); currentY = iY(y);}

   void lineTo(Graphics g, float x, float y) {
      int x1 = iX(x), y1 = iY(y);
      g.drawLine(currentX, currentY, x1, y1);
      currentX = x1; currentY = y1;
   }

   void drawArrow(Graphics g, float[] x, float[] y) {
      moveTo(x[0], y[0]);
      lineTo(g, x[1], y[1]);
      lineTo(g, x[2], y[2]);
      lineTo(g, x[3], y[3]);
      lineTo(g, x[1], y[1]);
   }

   public void paint(Graphics g) {
      float r = 40.0F;
      float[] x = {r, r, r - 2, r + 2}, y = {-7, 7, 0, 0};
      initgr();
      // Show coordinate axes:
      moveTo(30, 0); lineTo(g, 0, 0); lineTo(g, 0, 30);
      // Show initial arrow:
      drawArrow(g, x, y);
      float phi = (float) (2 * Math.PI / 3),
            c = (float) Math.cos(phi), s = (float) Math.sin(phi),
            r11 = c, r12 = s, r21 = -s, r22 = c;
      for (int j = 0; j < 4; j++) {
          float xNew = x[j] * r11 + y[j] * r21,
                yNew = x[j] * r12 + y[j] * r22;
          x[j] = xNew; y[j] = yNew;
      }
      // Arrow after rotation:
      drawArrow(g, x, y);
   }
}
```

The logical coordinates of the four relevant points of the arrow are stored in the arrays x and y, and the variables $r11$, $r12$, $r21$ and $r22$ denote the elements of the rotation matrix. When programming rotations, we should be careful with two points. First, with a constant angle φ, we should compute $\cos \varphi$ and $\sin \varphi$ only once, even though they occur twice in the rotation matrix. Second, a serious and frequently occurring error is modifying $x[j]$ too early, that is, while we still need the old value for the computation of $y[j]$, as the following, incorrect fragment shows:

```
x[j] = x[j] * r11 + y[j] * r21;  // ???
y[j] = x[j] * r12 + y[j] * r22;
```

There is no such problem if we use temporary variables *xNew* and *yNew* (although only the former is really required), as is done in the program.

Scaling

Suppose that we want to perform scaling with scale factors s_x for x and s_y for y and with point O remaining at its place; the latter is also expressed by referring to O as a *fixed point* or by a *scaling with reference to* O. This can obviously be written as.

$$\begin{cases} x' = s_x x \\ y' = s_y y \end{cases}$$

which can also be written as a very simple matrix multiplication:

$$[x' \quad y'] = [x \quad y] \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

There are some important special cases:

$s_x = s_y = -1$	gives a reflection about O;
$s_x = 1,\ s_y = -1$	gives a reflection about the x-axis;
$s_x = -1,\ s_y = 1$	gives a reflection about the y-axis.

Shearing

Consider the linear transformation given by

$$\begin{aligned} (1,0) &\rightarrow (1,0) \\ (0,1) &\rightarrow \ (a,1) \end{aligned}$$

Since the images of the unit vectors appear as the rows of the transformation matrix, we can write this transformation, known as *shearing*, as

$$[x' \quad y'] = [x \quad y] \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

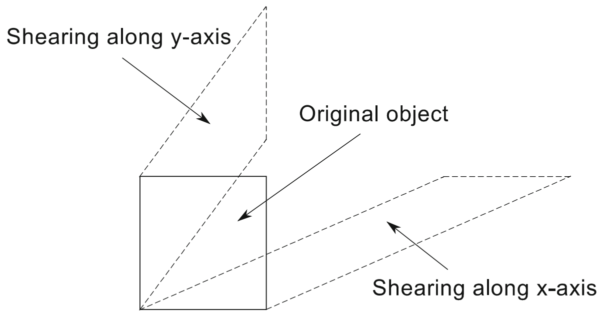


Fig. 3.3 Shearing effects (*dashed lines*) on a square object (*solid lines*)

or

$$\begin{cases} x' = x + ay \\ y' = y \end{cases}$$

This set of equations expresses that each point (x, y) moves a distance ay to the right, which has the effect of shearing along the x-axis, as illustrated in Fig. 3.3. We can use this transformation to turn regular characters into italic ones; for example, **L** becomes ***L***.

Shearing along the y-axis, also depicted in Fig. 3.3, can be similarly expressed as

$$[x' \quad y'] = [x \quad y] \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{cases} x' = x \\ y' = bx + y \end{cases}$$

3.3 Translations

Shifting all points in the xy-plane a constant distance in a fixed direction is referred to as a *translation*. This is another transformation, which we can write as:

$$x' = x + a$$
$$y' = y + b$$

We refer to the number pair (a, b) as the *shift vector*, or *translation vector*. Although this transformation is a very simple one, it is not linear, as we can easily see by the fact that the image of the origin $(0, 0)$ is (a, b), while this can only be the origin itself with linear transformations. Consequently, we cannot obtain the image (x', y') by multiplying (x, y) by a 2×2 transformation matrix T, which prevents us from combining such a matrix with other transformation matrices to obtain composite transformations. Fortunately, there is a solution to this problem as described in the following section.

3.4 Homogeneous Coordinates

To express all the transformations introduced so far as matrix multiplications in order to combine various transformation effects, we add one more dimension. As illustrated in Fig. 3.4, the extra dimension W makes any point $P = (x, y)$ of normal coordinates to have a whole family of homogeneous coordinate representations (wx, wy, w) for any value of w except 0. For example, (3, 6, 1), (0.3, 0.6, 0.1), (6, 12, 2), (12, 24, 4) and so on, represent the same point in two-dimensional space. Similarly, 4-tuples of coordinates represent points in three-dimensional space. When a point is mapped onto the $W = 1$ plane, in the form $(x, y, 1)$, it is said to be *homogenized*. In the above example, point (3, 6, 1) is homogenized, and the numbers 3, 6 and 1 are *homogeneous coordinates*.

In general, to convert a point from normal coordinates to homogeneous coordinates, add a new dimension to the right with value 1. To convert a point from homogeneous coordinates to normal coordinates, divide all the dimension values by the rightmost dimension value, and then discard the rightmost dimension.

Having introduced homogeneous coordinates, we are able to describe a translation by a matrix multiplication using a 3×3 instead of a 2×2 matrix. Using a shift vector (a, b), we can write the translation of Sect. 3.3 as the following matrix product:

$$[x' \quad y' \quad 1] = [x \quad y \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix} \tag{3.4}$$

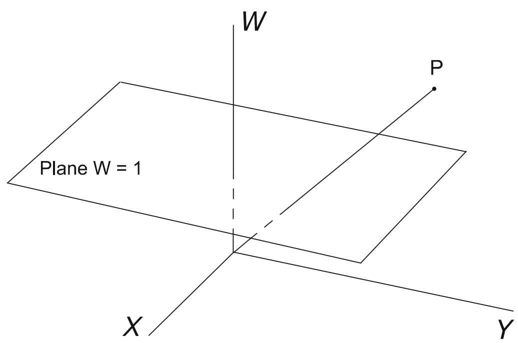


Fig. 3.4 A homogeneous coordinate system with the plane $W = 1$

Since we cannot multiply a 3 × 3 by a 2 × 2 matrix, we will also add a row and a column to linear transformation matrices if we want to combine these with translations (and possibly with other nonlinear transformations). These additional rows and columns simply consist of zeros followed by a one at the end. For example, we can use the following equation instead of (3.3) (of Sect. 3.2) for a rotation about O through the angle φ:

$$[x' \quad y' \quad 1] = [x \quad y \quad 1] \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

3.5 Inverse Transformations and Matrix Inversion

A linear transformation may or may not be reversible. For example, if we perform a rotation about the origin through an angle φ and let this follow by another rotation, also about the origin but through the angle $-\varphi$, these two transformations cancel each other out. Let us denote the rotation matrix of Eq. (3.3) (of Sect. 3.2) by R. It then follows that the inverse rotation, through the angle $-\varphi$ instead of φ, is described by the equation

$$[x' \quad y'] = [x \quad y]\, R^{-1}$$

where

$$R^{-1} = \begin{bmatrix} \cos(-\varphi) & \sin(-\varphi) \\ -\sin(-\varphi) & \cos(-\varphi) \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

The second equality in this equation is based on

$$\cos -\varphi = \cos\varphi$$
$$\sin -\varphi = -\sin\varphi$$

Matrix R^{-1} is referred to as the *inverse* of matrix R. In general, if a matrix A has an inverse, this is written A^{-1} and we have

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix, consisting of zero elements except for the main diagonal, which contains elements one. For example, in the case of a rotation through φ, followed by one through $-\varphi$, we have

$$RR^{-1} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The identity matrix clearly maps each point to itself, that is,

$$[x \quad y]I = [x \quad y]$$

Not all linear transformations are reversible. For example, the one that projects each point onto the x-axis is not. This transformation is described by

$$\begin{cases} x' = x \\ y' = 0 \end{cases}$$

which we can also write as

$$[x' \quad y'] = [x \quad y] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The 2×2 matrix in this equation has no inverse. This corresponds to the impossibility to reverse the linear transformation in question: since any two point $P_1(x, y_1)$ and $P_2(x, y_2)$ have the same image $P'(x, 0)$, it is impossible to find a unique point P of which P′ is the image.

A (square) matrix has an inverse if and only if its determinant is nonzero. For example, the determinant

$$\begin{vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{vmatrix}$$

is equal to $\cos\varphi \times \cos\varphi - (-\sin\varphi \times \sin\varphi) = \cos^2\varphi + \sin^2\varphi = 1$. Since this value is nonzero for any angle φ, the corresponding matrix

$$\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

has an inverse.

Exercise 3.4 shows how to compute the inverse of any 2×2 matrix that has a nonzero determinant. A useful application of this follows in Exercise 3.5.

3.6 Rotation About an Arbitrary Point

So far we have only performed rotations about the origin O. A rotation about any point other than O is not a linear transformation, since it does not map the origin onto itself. It can nevertheless be described by a matrix multiplication, provided we use homogeneous coordinates. A rotation about the point C(x_C, y_C) through the angle φ can be performed in three steps:

1. A translation from C to O, described by $[x' \quad y' \quad 1] = [x \quad y \quad 1]T^{-1}$, where

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_C & -y_C & 1 \end{bmatrix}$$

2. A rotation about O through the angle φ described by $[x' \quad y' \quad 1] = [x \quad y \quad 1]R_O$, where

$$R_O = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. A translation from O to C, described by $[x' \quad y' \quad 1] = [x \quad y \quad 1]T$, where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_C & y_C & 1 \end{bmatrix}$$

Note that we deliberately use the notations T^{-1} and T, since these two matrices are each other's inverse. This is understandable, since the operations of translating from C to O and then back from C to O cancel each other out.

The purpose of listing the above three matrices is that we can combine them by forming their product. Therefore, the desired rotation about point C through the angle φ can be described by

$$[x' \quad y' \quad 1] = [x \quad y \quad 1]R$$

where

$$R = T^{-1}R_O T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_C & -y_C & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_C & y_C & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ -x_C\cos\varphi + y_C\sin\varphi + x_C & -x_C\sin\varphi - y_C\cos\varphi + y_C & 1 \end{bmatrix} \tag{3.6}$$

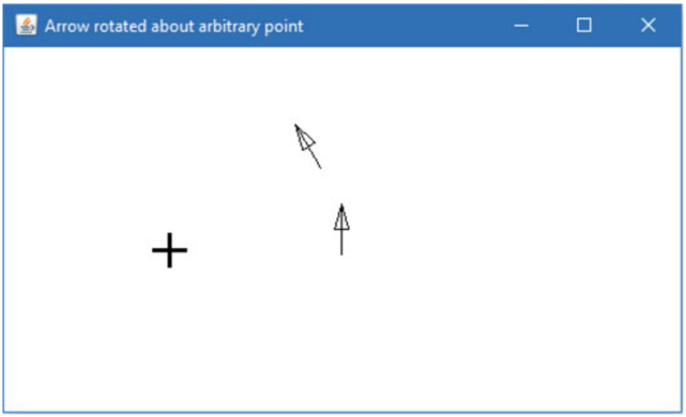


Fig. 3.5 Arrow before and after rotation through 30° about a point selected by the user

An Application

To see this general type of rotation in action, we will now discuss a program which rotates an arrow through 30° about a point selected by the user. Initially, one arrow, pointing vertically upward, appears in the center of the canvas. As soon as the user presses a mouse button, a second arrow appears. This is the image of the first one, resulting from a rotation through an angle of 30° about the cursor position. This position is displayed as a crosshair cursor in Fig. 3.5.

This action can be done repeatedly, in such a way that the most recently rotated arrow is again rotated when the user clicks, and this last rotation is performed about the most recently selected point. Since the rotation is counter-clockwise, the new arrow would have appeared below the old one in Fig. 3.5 if the user had selected a point to the right instead of to the left of the first arrow. Program *ArrowPt.java* shows how this rotation is computed.

```
// ArrowPt.java: Arrow rotated through 30 degrees
//    about a point selected by the user.
import java.awt.*;
import java.awt.event.*;

public class ArrowPt extends Frame {
   public static void main(String[] args) {
      new ArrowPt();
   }

   ArrowPt() {
      super("Arrow rotated about arbitrary point");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
```

```
      setSize(500, 300);
      add("Center", new CvArrowPt());
      setCursor(Cursor.getPredefinedCursor(Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvArrowPt extends Canvas {
   int centerX, centerY, currentX, currentY;
   float pixelSize, xP = 1e9F, yP, rWidth = 100.0F, rHeight = 100.0F;
   float[] x = {0, 0, -2, 2}, y = {-7, 7, 0, 0};

   CvArrowPt() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            xP = fx(evt.getX()); yP = fy(evt.getY());
            repaint();
         }
      });
   }

   void initgr() {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
   }

   int iX(float x) {return Math.round(centerX + x / pixelSize);}
   int iY(float y) {return Math.round(centerY - y / pixelSize);}
   float fx(int x) {return (x - centerX) * pixelSize;}
   float fy(int y) {return (centerY - y) * pixelSize;}

   void moveTo(float x, float y) {
      currentX = iX(x); currentY = iY(y);
   }

   void lineTo(Graphics g, float x, float y) {
      int x1 = iX(x), y1 = iY(y);
      g.drawLine(currentX, currentY, x1, y1);
      currentX = x1; currentY = y1;
   }
```

```
   void drawArrow(Graphics g, float[] x, float[] y) {
      moveTo(x[0], y[0]);
      lineTo(g, x[1], y[1]);
      lineTo(g, x[2], y[2]);
      lineTo(g, x[3], y[3]);
      lineTo(g, x[1], y[1]);
   }

   public void paint(Graphics g) {
      initgr();
      // Show initial arrow:
      drawArrow(g, x, y);
      if (xP > 1e8F)
         return;
      float phi = (float) (Math.PI / 6),
           c = (float) Math.cos(phi), s = (float) Math.sin(phi),
           r11 = c, r12 = s,
           r21 = -s, r22 = c,
           r31 = -xP * c + yP * s + xP, r32 = -xP * s - yP * c + yP;
      for (int j = 0; j < 4; j++) {
          float xNew = x[j] * r11 + y[j] * r21 + r31,
                yNew = x[j] * r12 + y[j] * r22 + r32;
          x[j] = xNew; y[j] = yNew;
      }
      // Arrow after rotation:
      drawArrow(g, x, y);
   }
}
```

In contrast to program *Arrow.java* of Sect. 3.2, this new program *ArrowPt.java* uses the 3×3 rotation matrix displayed in Eq. (3.6), as you can see in the fragment

```
float xNew = x[j] * r11 + y[j] * r21 + r31,
      yNew = x[j] * r12 + y[j] * r22 + r32;
```

The matrix elements r_{31} and r_{32} of the third row of the matrix depend on the point (x_P, y_P), selected by the user and acting as the center $C(x_C, y_C)$ in our previous discussion. As in program *Isotrop.java* of Sect. 1.3, the device coordinates of the selected point P are converted to logical coordinates by the methods *fx* and *fy*.

3.7 Changing the Coordinate System

In the preceding sections, we have used a fixed coordinate system and applied transformations to points given by their coordinates for that system, using certain computations. We can use exactly the same computations for a different purpose, leaving the points unchanged but changing the coordinate system. It is then important to bear in mind that the direction in which the coordinate system moves is opposite to that of the point movement. We can see this very clearly in the case of a translation. In Fig. 3.6a we have a normal translation with any point P (x, y) mapped to its image $P'(x', y')$, where

$$x' = x + a$$
$$y' = y + b$$

which can be written in matrix form as shown in Eq. (3.4) (of Sect. 3.4). In Fig. 3.6b we do not map the point P to another point but we express the position of this point in the $x'y'$-coordinate system, while it coordinates x and y are given. As you can see, a translation upward and to the right in (a), corresponds with a movement of the coordinate system downward and to the left in (b): these two directions are exactly each other's opposite. It follows that the inverse translation matrix would have applied if, in (b), we had moved the axes in the same direction as that of the point translation. The same principle applies to other transformations, such as rotations, for which the inverse of the transformation matrix exists. We will use this principle in the next section.

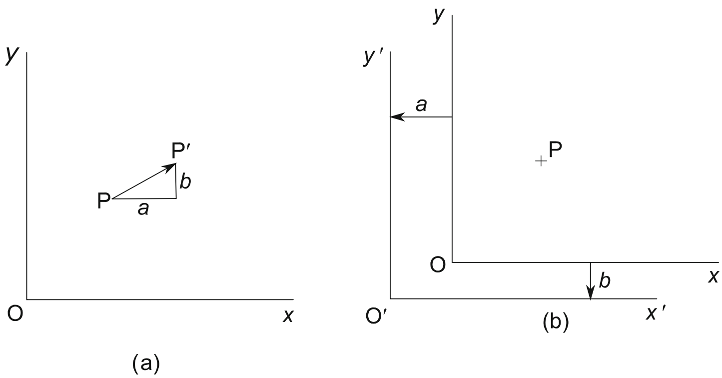


Fig. 3.6 (**a**) Translation and (**b**) change of coordinates

3.8 Rotations About 3D Coordinate Axes

Let us use a right-handed three-dimensional coordinate system, with the positive x-axis pointing toward us, the y-axis pointing to the right and the z-axis pointing upward, as shown in Fig. 3.7.

This figure also shows what we mean by rotations about the axes. A rotation about the z-axis through a given angle implies a rotation of all points in the xy-plane through that angle. For example, if this angle is 90°, the image of all points of the positive x-axis will be those of the positive y-axis. In the same way, a rotation about the x-axis implies a similar rotation of the yz-plane and a rotation about the y-axis implies a similar rotation of the zx-plane. Note that we deliberately write zx in that order: when we are dealing with the x-, y- and z-axes in a cyclic way, x follows z. It is important to remember this when we have to write down the transformation matrices for the rotations about the x-, y- and z-axis through the angle φ:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These matrices are easy to construct. Matrix R_z is derived in a trivial way from the well-known 2×2 rotation matrix of Eq. (3.3) (of Sect. 3.2). If you check this carefully, you will not find the other two matrices difficult. First, there is a 1 on the

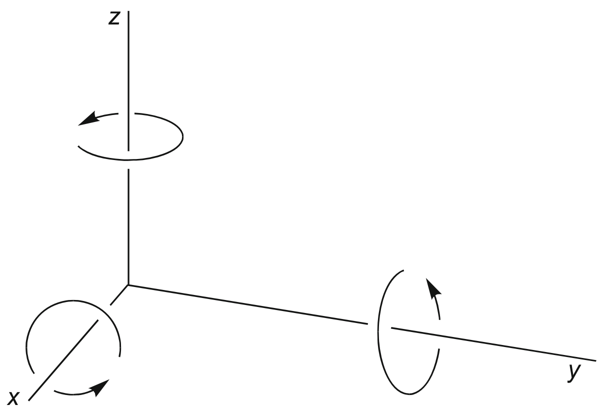


Fig. 3.7 Rotations about coordinate axes

main diagonal in the position that corresponds to the axis of rotation (1 for x, 2 for y and 3 for z). The other elements in the same row or column as this matrix element 1 are equal to 0. Second, we use the elements of the 2×2 matrix just mentioned for the remaining elements of the 3×3 matrices, beginning just to the right and below the element 1, if that is possible, If not, we remember that x follows z. For example, in R_y, the first element, cos φ, of this imaginary 2×2 matrix is placed in row 3 and column 3 because the element 1 has been placed in row 2 and column 2. Then, since we cannot place sin φ to the right of this element cos φ as we would like, we place it instead in column 1 of the same third row. In the same way, we cannot place –sin φ below cos φ, as it occurs in Eq. (3.3), so instead we put it in the first row of the same third column, and so on.

We should remember that the above matrices should be applied to row vectors. For example, using the above matrix R_x, we write

$$[x' \quad y' \quad z'] = [x \quad y \quad z]R_x$$

to obtain the image (x', y', z') of point (x, y, z) when the latter is subjected to a rotation about the x-axis through an angle φ.

3.9 Rotation About an Arbitrary Axis

To prepare for a three-dimensional rotation about an arbitrary axis, let us first perform such a rotation about an axis through the origin O. Actually, the rotation will take place about a vector, so that we can define its orientation, as illustrated by Fig. 3.8.

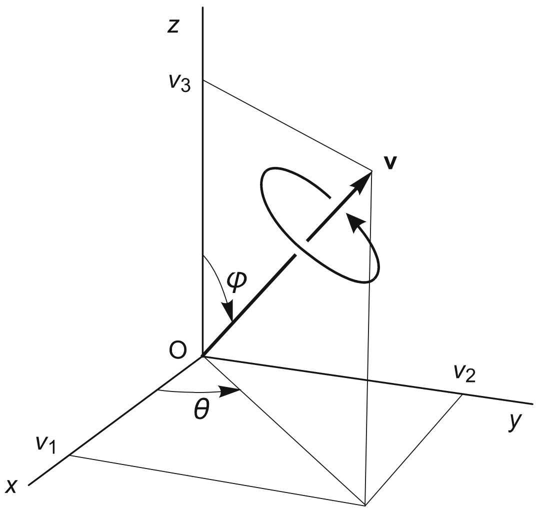


Fig. 3.8 Rotation about a vector starting in O

If a point is rotated about the vector **v** through a positive angle α, this rotation will be such that it corresponds to a movement in the direction of the vector in the same way as turning a (right-handed) screw corresponds to its forward movement.

Instead of its Cartesian coordinates (v_1, v_2, v_3), we will use the angles θ and φ to specify the direction of the vector **v**. The length of this vector is irrelevant to our present purpose. As you can see in Fig. 3.8, θ is the angle between the positive x-axis and the projection of **v** in the xy-plane and φ is the angle between the positive z-axis and vector **v**. If v_1, v_2 and v_3 are given and we want to find θ and φ, we can compute them in Java in the following way, writing *theta* for θ and *phi* for φ:

```
theta = Math.atan2(v2, v1);
phi = Math.atan2(Math.sqrt(v1 * v1 + v2 * v2), v3);
```

We will now derive a rather complicated 3×3 rotation matrix, which describes the rotation about the vector **v** through an angle α. First, we will change the coordinate system such that **v** will lie on the positive z-axis. This can be done in two steps:

1. A rotation about the z-axis, such that the horizontal component of **v** lies on the new x-axis.
2. A rotation of the coordinate system about the new y-axis though the angle φ.

As discussed at the end of the previous section, coordinate transformations require the inverses of the matrices that we would use for normal rotations of points. Referring to Sect. 3.8, we now have to use the following matrices R_z^{-1} and R_y^{-1} for the above steps 1 and 2, respectively:

$$R_z^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y^{-1} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

To combine these two coordinate transformations, we have to use the product of these two matrices. Before doing this matrix multiplication, let us first find the matrices of some more operations that are required.

Now that the new positive z-axis has the same direction as the vector **v**, the desired rotation about **v** through the angle α is also a rotation about the z-axis through that angle, so that, expressed in the new coordinates, we have the following rotation matrix:

$$R_{\mathbf{v}} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Although this may seem to be the final operation, we must bear in mind that we want to express the image point P′ of an original point P in terms of the original coordinate system. This implies that the first two of the above three transformations are to be followed by their inverse transformations, in reverse order. Therefore, in this order, we have to use the following two matrices after the three above:

$$R_y = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The resulting matrix R, to be used in the equation

$$[x' \quad y' \quad z'] = [x \quad y \quad z]R$$

to perform a rotation about $\mathbf{v}$ through the angle α, can now be found as follows:

$$R = R_z{}^{-1}R_y{}^{-1}R_{\mathbf{v}}R_yR_z = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where the matrix elements r_{ij} are rather complicated expressions in φ, θ and α. Before discussing how to compute these, let us first turn to the original problem, which was the same as the above, except that we want to use any point A(a_1, a_2, a_3) as the start point of vector $\mathbf{v}$. We can do this by performing, in this order:

- a translation that shifts the point A to the origin O;
- the desired rotation using the above matrix R;
- the inverse of the translation just mentioned.

As discussed in Sects. 3.4 and 3.6, we need to use homogeneous coordinates in order to describe translations by matrix multiplications. Since we use 3×3 matrices for linear transformations in three-dimensional space, we have to use 4×4 matrices in connection with these homogenous coordinates. Based on the coordinates a_1, a_2 and a_3 of the point A on the axis of rotation, the following matrix describes the translation from A to O:

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_1 & -a_2 & -a_3 & 1 \end{bmatrix}$$

After this translation, we perform the rotation about the vector **v**, which starts at point A, using the above matrix R, which we write as R^* after adding an additional row and column in the usual way:

$$R^* = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, we use a translation from O back to A:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix}$$

Writing R_{GEN} for the desired general (4×4) rotation matrix, we have

$$R_{\text{GEN}} = T^{-1}R^*T$$

Since R_{GEN} is a 4×4 matrix, we use it as follows:

$$[\,x' \quad y' \quad z' \quad 1\,] = [\,x \quad y \quad z \quad 1\,]R_{\text{GEN}}$$

Implementation

Since we are now dealing with points in three-dimensional space, let us begin with defining the following class to represent such points:

```
// Point3D.java: Representation of a point in 3D space.
class Point3D {
   float x, y, z;
   Point3D(double x, double y, double z) {
      this.x = (float) x; this.y = (float) y; this.z = (float) z;
   }
}
```

As we normally have a great many points that are to be rotated, it is worthwhile to compute the matrix R_{GEN} beforehand. Although this could be done numerically, it is also possible to make our program slightly faster by doing this symbolically, that is, by expressing all matrix elements r_{ij} of R_{GEN} in six constant values for the rotation: the angles φ, θ and α and the coordinates a_1, a_2 and a_3 of point A on the axis of rotation. Instead of writing these matrix elements here in the usual mathematical formulas (which would be quite complicated), we may as well immediately present the resulting Java code. This has been done in the class *Rotate*3*D* below.

The actual rotation is performed by the *rotate* method, which is called as many times as there are relevant points (usually the vertices of polyhedra). Prior to this, the method *initRotate* is called only once, to build the above matrix R_{GEN}. There are actually two methods *initRotate*, of which we can choose one as we like. The first accepts two points A and B to specify the directed axis of rotation AB and computes the angles θ and φ. The second accepts these two angles themselves instead of point B:

```
// Rota3D.java: Class used in other program files
//    for rotations about an arbitrary axis.
// Uses: Point3D (discussed above).
class Rota3D {
   static double r11, r12, r13, r21, r22, r23,
                 r31, r32, r33, r41, r42, r43;

/* The method initRotate computes the general rotation matrix

              | r11  r12  r13  0 |
        R =   | r21  r22  r23  0 |
              | r31  r32  r33  0 |
              | r41  r42  r43  1 |

   to be used as [x1  y1  z1  1] = [x  y  z  1] R
   by the method 'rotate'.
   Point (x1, y1, z1) is the image of (x, y, z).
   The rotation takes place about the directed axis
   AB and through the angle alpha.
*/
   static void initRotate(Point3D a, Point3D b, double alpha) {
      double v1 = b.x - a.x, v2 = b.y - a.y, v3 = b.z - a.z,
             theta = Math.atan2(v2, v1),
             phi = Math.atan2(Math.sqrt(v1 * v1 + v2 * v2), v3);
      initRotate(a, theta, phi, alpha);
   }

   static void initRotate(Point3D a, double theta, double phi,
         double alpha) {
      double cosAlpha, sinAlpha, cosPhi, sinPhi, cosTheta, sinTheta,
             cosPhi2, sinPhi2, cosTheta2, sinTheta2, c,
             a1 = a.x, a2 = a.y, a3 = a.z;
```

```
      cosPhi = Math.cos(phi); sinPhi = Math.sin(phi);
      cosPhi2 = cosPhi * cosPhi; sinPhi2 = sinPhi * sinPhi;
      cosTheta = Math.cos(theta); sinTheta = Math.sin(theta);
      cosTheta2 = cosTheta * cosTheta; sinTheta2 = sinTheta * sinTheta;
      cosAlpha = Math.cos(alpha); sinAlpha = Math.sin(alpha);
      c = 1.0 - cosAlpha;
      r11 = cosTheta2 * (cosAlpha * cosPhi2 + sinPhi2)
           + cosAlpha * sinTheta2;
      r12 = sinAlpha * cosPhi + c * sinPhi2 * cosTheta * sinTheta;
      r13 = sinPhi * (cosPhi * cosTheta * c - sinAlpha * sinTheta);
      r21 = sinPhi2 * cosTheta * sinTheta * c - sinAlpha * cosPhi;
      r22 = sinTheta2 * (cosAlpha * cosPhi2 + sinPhi2)
           + cosAlpha * cosTheta2;
      r23 = sinPhi * (cosPhi * sinTheta * c + sinAlpha * cosTheta);
      r31 = sinPhi * (cosPhi * cosTheta * c + sinAlpha * sinTheta);
      r32 = sinPhi * (cosPhi * sinTheta * c - sinAlpha * cosTheta);
      r33 = cosAlpha * sinPhi2 + cosPhi2;
      r41 = a1 - a1 * r11 - a2 * r21 - a3 * r31;
      r42 = a2 - a1 * r12 - a2 * r22 - a3 * r32;
      r43 = a3 - a1 * r13 - a2 * r23 - a3 * r33;
   }

   static Point3D rotate(Point3D p) {
      return new Point3D(
            p.x * r11 + p.y * r21 + p.z * r31 + r41,
            p.x * r12 + p.y * r22 + p.z * r32 + r42,
            p.x * r13 + p.y * r23 + p.z * r33 + r43);
   }
}
```

Note that the actual rotation of points is done very efficiently in the method *rotate* at the end of this class. This is important because it will be called for each relevant point of the object to be rotated. Remember, the more time-consuming method *initRotate* is called only once.

Let us now see this class *Rota*3*D* in action. Although it can be used for any rotation axis and any angle of rotation, it is used here only in a very simple way so that we can easily check the result, even without graphics output. As Fig. 3.9 shows, we have chosen the axis AB parallel to the diagonals 0–2 and 4–6 of the cube. The cube has a height 1 and point A has coordinates (0, 0, 2).

The following program uses the above classes *Point*3*D* and *Rota*3*D* to perform the rotation shown in Fig. 3.9:

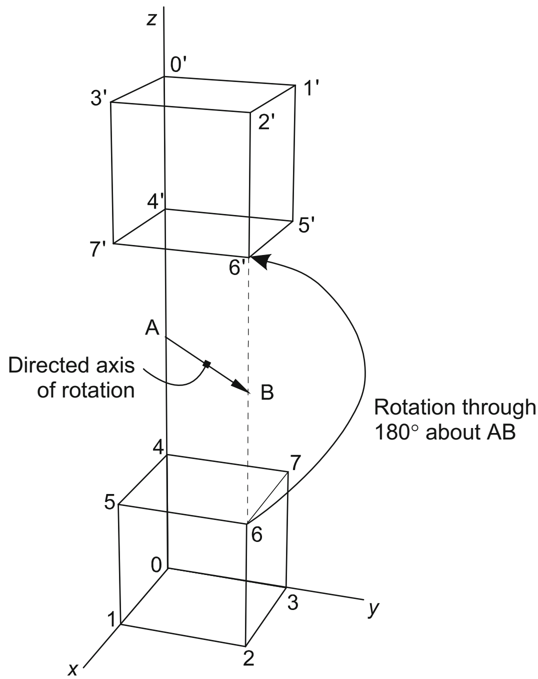


Fig. 3.9 Rotation of a cube about the axis AB

```
// Rota3DTest.java: Rotating a cube about an axis
//    parallel to a diagonal of its top plane.
//    Uses: Point3D, Rota3D (discussed above).
public class Rota3DTest {
   public static void main(String[] args) {
      Point3D a = new Point3D(0, 0, 2), b = new Point3D(1, 1, 2);
      double alpha = Math.PI;
      // Specify AB as directed axis of rotation
      // and alpha as the rotation angle:
      Rota3D.initRotate(a, b, alpha);
      // Vertices of a cube; 0, 1, 2, 3 at the bottom,
      // 4, 5, 6, 7 at the top. Vertex 0 at the origin O:
      Point3D[] v = {
            new Point3D(0, 0, 0), new Point3D(1, 0, 0),
            new Point3D(1, 1, 0), new Point3D(0, 1, 0),
            new Point3D(0, 0, 1), new Point3D(1, 0, 1),
            new Point3D(1, 1, 1), new Point3D(0, 1, 1)};
      System.out.println(
            "Cube rotated through 180 degrees about line AB,");
      System.out.println("where A = (0, 0, 2) and B = (1, 1, 2)");
      System.out.println("Vertices of cube:");
      System.out.println("    Before rotation    After rotation");
```

```
      for (int i = 0; i < 8; i++) {
         Point3D p = v[i];
         // Compute P1, the result of rotating P:
         Point3D p1 = Rota3D.rotate(p);
         System.out.println(i + ":  " +
            p.x + "  " + p.y + "  " + p.z + "        " +
            f(p1.x) + "  " + f(p1.y) + "  " + f(p1.z));
      }
   }

   static double f(double x) {return Math.abs(x) < 1e-10 ? 0.0 : x;}
}
```

Since we have not yet discussed how to produce perspective views, we produce only text output in this program, as listed below:

```
Cube rotated through 180 degrees about line AB,
where A = (0, 0, 2) and B = (1, 1, 2)
Vertices of cube:
    Before rotation    After rotation
0:  0.0 0.0 0.0        0.0 0.0 4.0
1:  1.0 0.0 0.0        0.0 1.0 4.0
2:  1.0 1.0 0.0        1.0 1.0 4.0
3:  0.0 1.0 0.0        1.0 0.0 4.0
4:  0.0 0.0 1.0        0.0 0.0 3.0
5:  1.0 0.0 1.0        0.0 1.0 3.0
6:  1.0 1.0 1.0        1.0 1.0 3.0
7:  0.0 1.0 1.0        1.0 0.0 3.0
```

Exercises

3.1. In Sect. 3.2 we discussed scaling with reference to the origin O, that is, with O as a fixed point. It is also possible to use a different fixed point, say, C(x_C, y_C), but, for such a scaling in two-dimensional space, we need a 3 × 3 matrix M (and homogenous coordinates), writing

$$[x' \quad y' \quad 1] = [x \quad y \quad 1]M$$

Using scale factors s_x and s_y for x and y again, find this matrix M.

Hint: You can perform a translation from C to O, followed by a scaling as discussed in Sect. 3.2, but described by a 3 × 3 matrix, followed by a translation back from O to C. Alternatively, you can start with the following system of equations, which shows very clearly what actually happens:

$$\begin{cases} x' - x_C = s_x(x - x_C) \\ y' - y_C = s_y(y - y_C) \end{cases}$$

3.2. Describe scaling in three-dimensional space with reference to a point C and three scale factors s_x, s_y and s_z. Find the 4 × 4 matrix (similar to matrix M of Exercise 3.1) for this transformation.

3.3. How can you apply shearing with reference to a point other than O (such that this point will remain at its place)? Write a program that draws a square and an approximated circle, both before and after shearing, setting the constant a used at the end of Sect. 3.2 equal to 0.5. Apply shearing to these two figures with reference to their centers; in other words, the center of the square will remain at its place and so will the center of the circle.

3.4. (This exercise prepares for the next one.) If the determinant $D = a_{11}a_{22} - a_{12}a_{21}$ of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is nonzero, then the matrix

$$A^{-1} = \begin{bmatrix} \dfrac{a_{22}}{D} & -\dfrac{a_{12}}{D} \\ -\dfrac{a_{21}}{D} & \dfrac{a_{11}}{D} \end{bmatrix}$$

is the inverse of A, that is, $AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Prove this.

3.5. Our way of testing whether a given point P lies in a triangle ABC, discussed in Sect. 2.5, resulted in the method *insideTriangle*, which assumes that A, B and C are counter-clockwise. Develop a different method for the same purpose, based on the vectors $\mathbf{a} = (a_1, a_2) = \mathbf{CA}$ and $\mathbf{b} = (b_1, b_2) = \mathbf{CB}$ (see Fig. 2.6 in Sect. 2.3). Let us write

$$\mathbf{CP} = \mathbf{p} = (p_1, p_2) = \lambda\mathbf{a} + \mu\mathbf{b}$$

or, in the form of a matrix product,

$$[p_1 \quad p_2] = [\lambda \quad \mu] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \tag{3.7}$$

Since we saw in Exercise 3.4 how to compute the inverse of a 2 × 2 matrix, we can now compute λ and μ as follows:

$$[\lambda \quad \mu] = [p_1 \quad p_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^{-1} \tag{3.8}$$

The point P lies in triangle ABC (or on one of its edges) if and only if $\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu \leq 1$. Write the class *TriaTest*, which we can use as follows:

```
Point2D a, b, c, p;                                          ...
TriaTest tt = new TriaTest(a, b, c);
if (tt.area2() != 0 && tt.insideTriangle(p)) ...
   // Point P within triangle ABC.
```

As in Sect. 2.3, the method *area*2 returns twice of the area of triangle ABC, preceded by a minus sign if ABC is clockwise. This return value is also equal to the determinant of the 2×2 matrix in Eq. (3.7). We must not call the *TriaTest* method *insideTriangle* if this determinant is zero, since in that case the inverse matrix of Eq. (3.8) does not exist (see Exercise 3.4).

Chapter 4
Classic 2D Algorithms

Although programming is a creative activity, we can sometimes benefit from well-known algorithms, that provide more elegant solutions and optimizations than those we would have been able to invent ourselves. This is no different in computer graphics. This chapter presents several well-known graphics algorithms for (a) computing the coordinates of pixels that comprise lines and circles, (b) clipping lines and polygons, and (c) drawing smooth curves. These are the most primitive operations in computer graphics and should be executed as fast as possible. Therefore, the algorithms in this chapter ought to be optimized to avoid time-consuming executions, such as multiplication, division, and operations on floating point numbers.

4.1 Bresenham Line Drawing

Although, in Java, we can simply use the method *drawLine* to draw straight lines, it would be unsatisfactory if we had no idea how this method works. We will now discuss how to draw a straight line given the coordinates of its two endpoints by drawing appropriate individual pixels on the screen. We will be using integer coordinates, but when discussing the slope of a line it would be very inconvenient if we had to use a y-axis pointing downward, as is the case with the Java device coordinate system. Therefore, following the mathematical convention, the positive y-axis will point upward in our discussion.

Unfortunately, Java lacks a method with the sole purpose of drawing a pixel on the screen, so that we define the following rather strange method to achieve this:

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2_4

```
void putPixel(Graphics g, int x, int y) {
   g.drawLine(x, y, x, y);
}
```

Recalling several other Java methods we learned in Chap. 1, we could also use

```
g.fillRect(x, y, 1, 1)
```

to achieve the same result, that is, drawing a single pixel at (x, y).

We will now develop a *drawLine* method of the form

```
void drawLine(Graphics g, int xP, int yP, int xQ, int yQ) {
   ...
}
```

which only uses the above *putPixel* method for the actual graphics output.

Figure 4.1 shows a line segment with endpoints P(1, 1) and Q(12, 5), as well as the pixels that we have to compute to approximate this line. To draw this line, we then write

```
drawLine(g, 1, 1, 12, 5);
```

Let us first solve the problem for situations such as Fig. 4.1, in which point Q lies to the right and not lower than point P. More precisely, we will be considering the special case

$$
\begin{aligned}
&x_P < x_Q \\
&y_P \le y_Q \\
&y_Q - y_P \le x_Q - x_P
\end{aligned} \tag{4.1}
$$

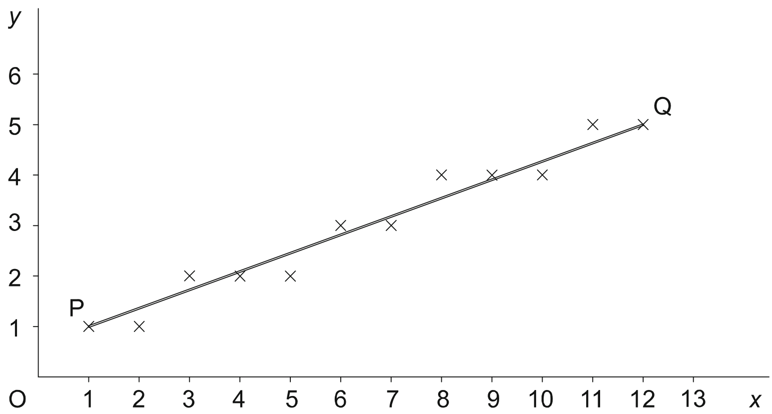


Fig. 4.1 Grid points approximating a line segment

where the last condition expresses that the angle of inclination of line PQ is not greater than 45°. We then wish to find exactly one integer y for each of the integers

$$x_P, x_P + 1, x_P + 2, \ldots, x_Q$$

Except for the first and the last (which are given as y_P and y_Q) the most straightforward way of computing these y-coordinates is by using the slope

$$m = \frac{y_Q - y_P}{x_Q - x_P} \tag{4.2}$$

so that, for each of the given integer x-coordinates, we can find the desired integer y-coordinate by rounding the value

$$y_{\text{exact}} = y_P + m(x - x_P)$$

to the nearest integer. Since two such successive x-coordinates differ by 1, the corresponding difference of two successive values of y_{exact} is equal to m. Figure 4.2 shows this interpretation of m, as well as that of the 'error'

$$d = y_{exact} - y \tag{4.3}$$

Since d is the error we make by rounding y_{exact} to the nearest integer, we can require d to satisfy the following condition:

$$-0.5 < d \leq +0.5 \tag{4.4}$$

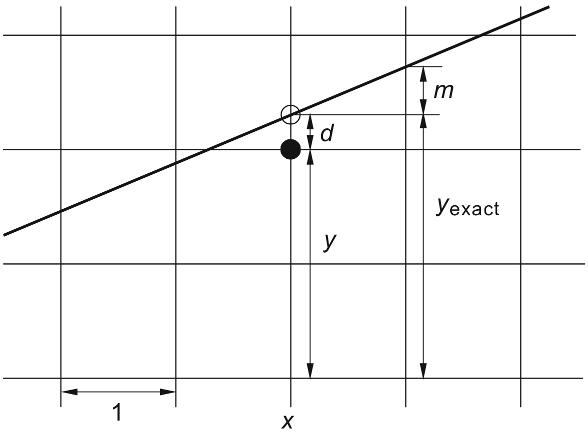


Fig. 4.2 Slope m and error d

The following first version of the desired *drawLine* method is based on this observation:

```
void drawLine1(Graphics g, int xP, int yP, int xQ, int yQ) {
   int x = xP, y = yP;
   float d = 0, m = (float)(yQ - yP)/(float)(xQ - xP);
   for (;;) {
      putPixel(g, x, y);
      if (x == xQ) break;
      x++;
      d += m;
      if (d >= 0.5){y++; d--;}
   }
}
```

This version is easy to understand if we pay attention to Fig. 4.2. Since the first call to *putPixel* applies to point P, we begin with the error $d = 0$. In each step of the loop, x is increased by 1. Assuming for the moment that y will not change, it follows from Eq. (4.3) that the growth of d will be the same as that of y_{exact}, which explains why d is increased by m. We then check the validity of this assumption in the following if-statement. If d has become greater than 0.5, this violates Eq. (4.4), so that we apply a correction, consisting of increasing y and decreasing d by 1. The latter action makes Eq. (4.4) valid again. By doing these two actions at the same time, Eq. (4.3) also remains valid.

We use *drawLine*1 as a basis for writing a faster version, which no longer uses type *float*. This is possible because the slope variable m represents a rational number, that is, an integer numerator divided by an integer denominator, as Eq. (4.2) shows. Since the other float variable, d, starts with the value zero and is altered only by adding m and -1 to it, it is also a rational number. In view of both the denominator $x_Q - x_P$ of these rational numbers and the constant 0.5 used in the if-statement, we will apply the scaling factor

$$c = 2\left(x_Q - x_P\right)$$

to m, to d and to this constant 0.5, introducing the *int* variables M and D instead of the *float* variables m and d. We will also use the Δx instead of the constant 0.5. These new values M, D and Δx are c times as large as m, d and 0.5, respectively:

$$\begin{aligned} M &= cm = 2\left(y_Q - y_P\right) \\ D &= cd \\ \Delta x &= x_Q - x_P = c \times 0.5 \end{aligned}$$

In this way we obtain the following integer version, which is very similar to the previous one and equivalent to it but faster. In accordance with the Java naming conventions for variables, we write m and d again instead of M and D, and dx instead of Δx:

```
void drawLine2(Graphics g, int xP, int yP, int xQ, int yQ) {
   int x = xP, y = yP, d = 0, dx = xQ - xP, c = 2 * dx,
       m = 2 * (yQ - yP);
   for (;;) {
      putPixel(g, x, y);
      if (x == xQ) break;
      x++;
      d += m;
      if (d >= dx){y++; d -= c;}
   }
}
```

Having dealt with this special case, with points P and Q satisfying Eq. (4.1), we now turn to the general problem, without any restrictions with regard to the relative positions of P and Q. To solve this, we have several symmetric cases to consider. As long as

$$| y_Q - y_P | \leq | x_Q - x_P |$$

we can again use x as the independent variable, that is, we can increase or decrease this variable by one in each step of the loop. In the opposite case, with lines that have an angle of inclination greater than 45°, we have to interchange the roles of x and y to prevent the selected pixels from lying too far apart. All this is realized in the general line-drawing method *drawLine* below. We can easily verify that this version plots exactly the same points approximating line PQ as version *drawLine*2 if the coordinates of P and Q satisfy (4.1).

```
void drawLine(Graphics g, int xP, int yP, int xQ, int yQ) {
   int x = xP, y = yP, d = 0, dx = xQ - xP, dy = yQ - yP,
      c, m, xInc = 1, yInc = 1;
   if (dx < 0){xInc = -1; dx = -dx;}
   if (dy < 0){yInc = -1; dy = -dy;}
   if (dy <= dx) {
      c = 2 * dx; m = 2 * dy;
      if (xInc < 0) dx++;
      for (;;) {
         putPixel(g, x, y);
         if (x == xQ) break;
         x += xInc;
         d += m;
         if (d >= dx){y += yInc; d -= c;}
      }
   }
```

```
   else {
      c = 2 * dy; m = 2 * dx;
      if (yInc < 0) dy++;
      for (;;) {
         putPixel(g, x, y);
         if (y == yQ) break;
         y += yInc;
         d += m;
         if (d >= dy){x += xInc; d -= c;}
      }
   }
}
```

Just before executing one of the above two for-statements, an if-statement is executed in the cases that *x* or *y* decreases instead of increases, which is required to guarantee that drawing PQ always plots exactly the same points as drawing QP.

The idea of drawing sloping lines by means of only integer variables was first realized by *Bresenham*; his name is therefore associated with this algorithm.

The above *drawLine* method is very easy to use if we are dealing with the floating-point logical-coordinate system used so far. For example, in Sect. 5, there will be a program *ClipPoly.java*, in which the following method occurs:

```
void drawLine(Graphics g, float xP, float yP, float xQ, float yQ) {
   g.drawLine(iX(xP), iY(yP), iX(xQ), iY(yQ));
}
```

Here our own method *drawLine* calls the Java method *drawLine* of the class *Graphics*. If, instead, we want to use our own method *drawLine* with four *int* arguments, listed above, we can replace the last three program lines with

```
void drawLine(Graphics g, float xP, float yP, float xQ, float yQ) {
   drawLine(g, iX(xP), iY(yP), iX(xQ), iY(yQ)); // int coordinates
}
```

provided that we also add the method *putPixel*, listed at the beginning of this section. This may at first be confusing because there are now two *drawLine* methods of our own. As the comment after the above call to *drawLine* indicates, the Java compiler will select our Bresenham method *drawLine* because there is an argument *g*, followed by four arguments of type *int*. It is also interesting to note that the direction of the positive *y*-axis in our discussion causes no practical problem.

4.2 Doubling the Line-Drawing Speed

As one of the primitive graphics operations, line drawing should be performed as rapidly as possible. In fact, graphics hardware is typically benchmarked by the speed in which it generates lines. Bresenham's line algorithm is simple and efficient in generating lines. The algorithm works incrementally by computing the position of the next pixel to be drawn. Hence it iterates as many times as the number of pixels in the line it generates. The double-step line-drawing algorithm by Rokne, Wyvill, and Wu [22] aims at reducing the number of iterations by half, by computing the positions of the next *two* pixels.

Let us again start with the lines within the slope range of [0, 1] and consider the general case of any slopes later (as Exercise 4.2). For a line PQ, starting from P, we increment the x coordinate by two pixels instead of one as in Bresenham's algorithm. All the possible positions of the two pixels in the above slope range form four patterns, expressed in a 2×2 mesh illustrated in Fig. 4.3. It has been mathematically proven that patterns 1 and 4 would never occur on the same line, implying that a line would possibly involve patterns 1, 2 and 3, or patterns 2, 3 and 4, depending the slope of the line. The lines within the slope range of [0, ½) involve patterns 1, 2, and 3 (an example depicted as a solid line in Fig. 4.4), and lines within the slope range of (½, 1] involve patterns 2, 3 and 4 (an example as a dotted line). At the exact slope of ½, the line involves either pattern 2 or 3, not 1 or 4.

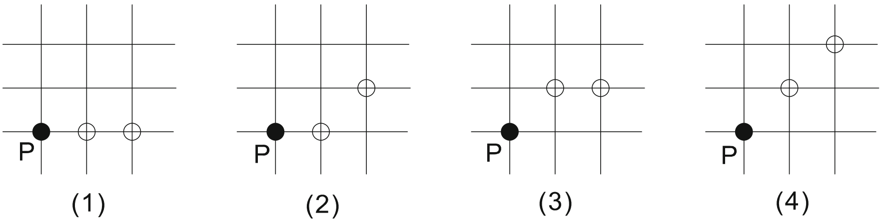


Fig. 4.3 Four double-step patterns when $0 \leq$ slope ≤ 1

Fig. 4.4 Choice of patterns based on initial error d and slope m

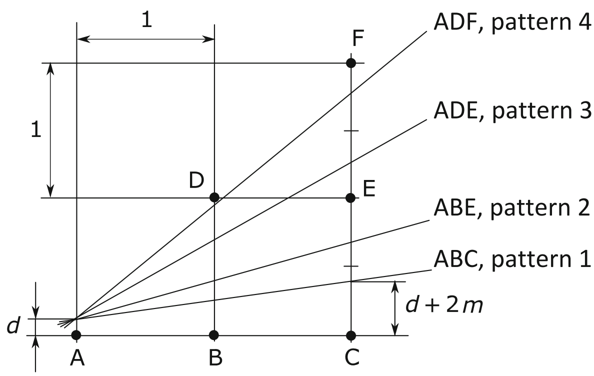


Figure 4.4 shows four sloping lines, all passing through the same point at a distance d above the point A. By approximating this common point of the four lines we make an error $d = y_{\text{exact}} - y$, which should not be greater than 0.5.

Since the slope of a line is equal to m, the exact y-coordinate of that line is equal to $d + m$ at its point of intersection with BD, and $d + 2\,m$ at its point of intersection with CF, as indicated in Fig. 4.4 for the lowest of the four lines. With these two points of intersection lying closer to BC than to DE, it is clear that this lowest sloping line should be approximated by the points A, B, C, that is, by pattern 1 of Fig. 4.3. Thus we have

if $d + 2\,m < 0.5$, we use pattern 1 (ABC).

Otherwise, we use point E instead of C if E is the best approximation of the point where the sloping line intersects CF, that is, if $0.5 \le d + 2\,m \le 1.5$. However, we should now also pay attention to the point where the sloping line intersects BD. Comparing $d + m$ with 0.5 determines whether B or D should be taken. More precisely,

if $0.5 \le d + 2\,m < 1.5$ (so that point E is to be used), we choose B or D as follows:
  if $d + m < 0.5$, we use pattern 2 (ABE);
  if $d + m \ge 0.5$, we use pattern 3 (ADE).

Finally, there is this remaining case:

if $d + 2\,m \ge 1.5$, we use pattern 4 (ADF).

As in the previous section when discussing Bresenham's algorithm, we begin with a preliminary method that still uses floating-point variables to make it easier to understand, so it is not yet optimized for speed. This version also works for lines drawn from right to left, that is, when $x_Q < x_P$, as well as for lines with negative slopes. However, the absolute value of the slope should not be greater than 1. So *doubleStep*1 applies only to endpoints P and Q that satisfy the following conditions:

$$x_Q \ne x_P$$
$$| y_Q - y_P | \le | x_Q - x_P |$$

It is wise to begin with the simplest case

$$x_P < x_Q$$
$$0 \le y_Q - y_P \le x_Q - x_P$$

when you read the following code for the first time:

```
void doubleStep1(Graphics g, int xP, int yP, int xQ, int yQ) {
   int dx, dy, x, y, yInc;
   if (xP >= xQ) {
      if (xP == xQ) // Not allowed because we divide by (dx = xQ - xP)
         return;
      // xP > xQ, so swap the points P and Q
      int t;
      t = xP; xP = xQ; xQ = t;
      t = yP; yP = yQ; yQ = t;
    }
    // Now xP < xQ
    if (yQ >= yP){yInc = 1; dy = yQ - yP;} // Normal case, yP < yQ
    else         {yInc = -1; dy = yP - yQ;}
    dx = xQ - xP; // dx > 0, dy > 0
    float d = 0,                  // Error d = yexact - y
        m = (float)dy/(float)dx; // m <= 1, m = |slope|
    putPixel(g, xP, yP);
    y = yP;
    for (x=xP; x<xQ-1;) {
       if (d + 2 * m < 0.5) { // Pattern 1:
          putPixel(g, ++x, y);
          putPixel(g, ++x, y);
          d += 2 * m; // Error increases by 2m, since y remains
                      // unchanged and yexact increases by 2m
       }
       else
       if (d + 2 * m < 1.5) { // Pattern 2 or 3
          if (d + m < 0.5) { // Pattern 2
             putPixel(g, ++x, y);
             putPixel(g, ++x, y += yInc);
             d += 2 * m - 1; // Because of ++y, the error is now
                             // 1 less than with pattern 1
          }
          else { // Pattern 3
             putPixel(g, ++x, y += yInc);
             putPixel(g, ++x, y);
             d += 2 * m - 1; // Same as pattern 2
          }
       }
       else { // Pattern 4:
          putPixel(g, ++x, y += yInc);
          putPixel(g, ++x, y += yInc);
          d += 2 * m - 2; // Because of y += 2, the error is now
                          // 2 less than with pattern 1
       }
    }
```

```
      if (x < xQ)  // x = xQ - 1
         putPixel(g, xQ, yQ);
}
```

Before the above for-loop is entered, there is a call to *putPixel* for $x = x_{\mathrm{P}}$. The loop terminates as soon as the test

$$x < x_{\mathrm{Q}} - 1$$

fails. This test is executed for the following values of x:

$$x_{\mathrm{P}}, x_{\mathrm{P}} + 2, x_{\mathrm{P}} + 4, \text{and so on.}$$

If it succeeds, *putPixel* is called for the next two values of x, not for the value of x used in the test. There are two cases to consider. If $x_{\mathrm{Q}} - x_{\mathrm{P}}$ is even, the test still succeeds when $x = x_{\mathrm{Q}} - 2$, and *putPixel* is executed for both $x = x_{\mathrm{Q}} - 1$ and $x = x_{\mathrm{Q}}$, after which we are done and the next test, with $x = x_{\mathrm{Q}}$, fails. On the other hand, if $x_{\mathrm{Q}} - x_{\mathrm{P}}$ is odd, the test succeeds when $x = x_{\mathrm{Q}} - 3$, and *putPixel* is called for both $x = x_{\mathrm{Q}} - 2$ and $x = x_{\mathrm{Q}} - 1$, after which the test fails with $x = x_{\mathrm{Q}} - 1$. Then, after loop termination, the remaining call to *putPixel* is executed in the following if-statement:

```
if (x < xQ)
   putPixel(g, xQ, yQ);
```

We will now derive a fast, integer version from the above method *doubleStep*1. Let us use the notation

$$\Delta x = x_{\mathrm{Q}} - x_{\mathrm{P}}$$
$$\Delta y = y_{\mathrm{Q}} - y_{\mathrm{P}}$$

so that slope $m = \Delta y / \Delta x$. We now want to introduce an *int* variable v, in such a way that the test

$$d + 2m < 0.5 \tag{4.5}$$

reduces to

$$v < 0 \tag{4.6}$$

To achieve this, we start writing (4.5) as $d + 2\,m - 0.5 < 0$ and, since this inequality contains the two fractions $m = \Delta y / \Delta x$ and 0.5, we multiply both sides of it by $2\Delta x$, obtaining

$$2d\Delta x + 4\Delta y - \Delta x < 0$$

Therefore, instead of the floating-point error variable $d = y_{\text{exact}} - y$, we will use the integer variable v, which relates to d as follows:

$$v = 2d\Delta x + 4\Delta y - \Delta x \tag{4.7}$$

We can now replace the test (4.5) with the more efficient one (4.6). It follows from (4.7) that increasing d by 2 m is equivalent to increasing v by $2\Delta x \times 2\,m$, which is equal to $4\Delta y$. This explains both the test $v < 0$ and adding $dy4$ to v in the for-loop below. These operations on the variable v are marked by comments of the form *// Equivalent to …*, as are some others, which are left to the reader to verify.

```
void doubleStep2(Graphics g, int xP, int yP, int xQ, int yQ) {
   int dx, dy, x, y, yInc;
   if (xP >= xQ) {
      if (xP == xQ) // Not allowed because we divide by (dx = xQ - xP)
         return;
      int t;    // xP > xQ, so swap the points P and Q
      t = xP; xP = xQ; xQ = t;
      t = yP; yP = yQ; yQ = t;
   }
   // Now xP < xQ
   if (yQ >= yP){yInc = 1; dy = yQ - yP;}
   else         {yInc = -1; dy = yP - yQ;}
   dx = xQ - xP;
   int dy4 = dy * 4, v = dy4 - dx, dx2 = 2 * dx, dy2 = 2 * dy,
      dy4Minusdx2 = dy4 - dx2, dy4Minusdx4 = dy4Minusdx2 - dx2;
   putPixel(g, xP, yP);
   y = yP;
   for (x=xP; x<xQ-1;) {
      if (v < 0) {                    // Equivalent to d + 2 * m < 0.5
         putPixel(g, ++x, y);         // Pattern 1
         putPixel(g, ++x, y);
         v += dy4;                    // Equivalent to d += 2 * m
      }
      else
      if (v < dx2) {                  // Equivalent to d + 2 * m < 1.5
         // Pattern 2 or 3
         if (v < dy2) {               // Equivalent to d + m < 0.5
            putPixel(g, ++x, y);      // Pattern 2
            putPixel(g, ++x, y += yInc);
            v += dy4Minusdx2;         // Equivalent to d += 2 * m - 1
         }
         else {
            putPixel(g, ++x, y += yInc); // Pattern 3
            putPixel(g, ++x, y);
            v += dy4Minusdx2;         // Equivalent to d += 2 * m - 1
         }
      }
```

```
        else {
            putPixel(g, ++x, y += yInc); // Pattern 4
            putPixel(g, ++x, y += yInc);
            v += dy4Minusdx4;       // Equivalent to d += 2 * m - 2
        }
    }
    if (x < xQ)
        putPixel(g, xQ, yQ);
}
```

Remember, the above method *doubleStep*2 works only if $|y_Q - y_P| \le |x_Q - x_P|$. In Exercise 4.2 the reader is to generalize the method to work for any line and add a simple interface to allow the user to enter lines.

Like Bresenham's line algorithm, the above double-step algorithm computes on integers only. For long lines, it is supposed to outperform Bresenham's algorithm by nearly two folds (to be evaluated in Exercise 4.3). One can further optimize the algorithm to achieve another two folds of speed-up by taking advantage of the symmetry around the midpoint of the given line. We leave this as an exercise for interested readers. The double-step algorithm can in fact be generalized to draw circles, which will be left for users' own exploration by consulting the papers by Wu and Rokne [25] and by Rokne, Wyvill, and Wu [22] listed in Bibliography at the end of the book. Adapting Bresenham's algorithm for circles will be the topic of the next section.

4.3 Circle Drawing

In this section we will ignore the normal way of drawing a circle in Java by a call of the form

```
g.drawOval(xC - r, yC - r, 2 * r, 2 * r);
```

since it is our goal to construct such a circle, with center $C(x_C, y_C)$ and radius r ourselves, where the coordinates of C and the radius are given as integers.

If speed is not a critical factor, we can apply the method *drawLine* to a great many neighboring points (x, y), computed as

$$x = x_C + r \cos \varphi$$
$$y = y_C + r \sin \varphi$$

where

$$\varphi = i \times \frac{2\pi}{n} \quad (i = 0, 1, 2, \ldots, n - 1)$$

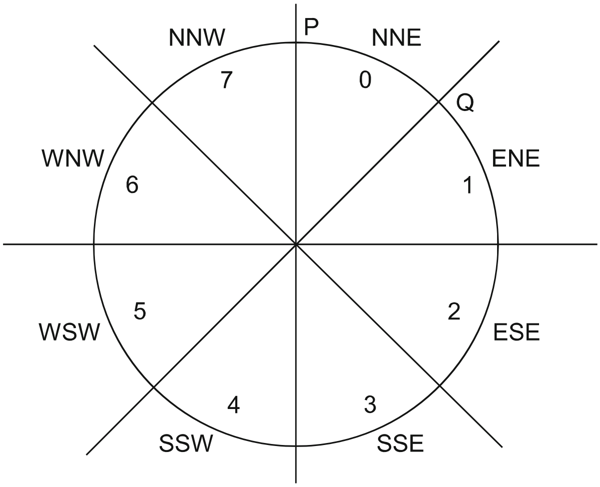


Fig. 4.5 Arc PQ of a circle with origin O and radius r

for some large value of n. Instead of the above two ways of drawing a circle, we will develop a method of the form

```
void drawCircle(Graphics g, int xC, int yC, int r) {
    ...
}
```

which uses only the method *putPixel* of the previous section as a graphics 'primitive', and which is an implementation of Bresenham's algorithm for circles. The circle drawn in this way will be exactly the same as that produced by the above call to *drawOval*. In both cases, x will range from $x_C - r$ to $x_C + r$, including these two values, so that $2r + 1$ different values of x will be used.

As in the previous sections, we begin with a simple case: we use the origin of the coordinate system as the center of the circle, and, dividing the circle into eight arcs of equal length, we restrict ourselves to one of these, the arc PQ. The points P(0, r) and Q($r/\sqrt{2}$, $r/\sqrt{2}$) are shown in Fig. 4.5.

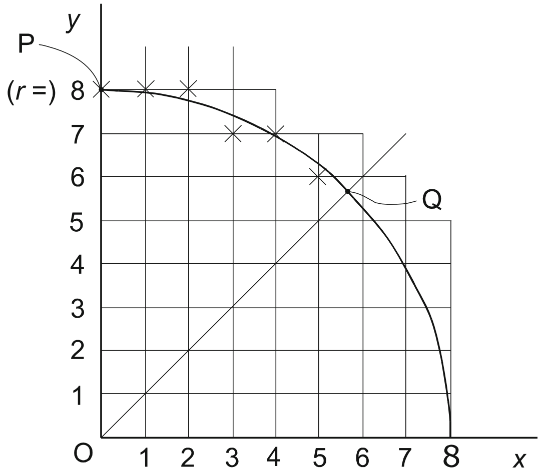
The equation of this circle is

$$x^2 + y^2 = r^2 \tag{4.8}$$

Figure 4.6 shows the situation, including the grid of pixels, for the case $r = 8$. Beginning at the top at point P, with $x = 0$ and $y = r$, we will use a loop in which we increase x by 1 in each step; as in the previous section, we need some test to decide whether we can leave y unchanged. If not, we have to decrease y by 1.

Since, just after increasing x by one, we have to choose between (x, y) and (x, $y - 1$), we could simply compute both

Fig. 4.6 Pixels that approximate the arc PQ



$$x^2 + y^2 \quad \text{and} \quad x^2 + (y-1)^2$$

to see which lies closer to r^2. Note that this can be done by using only integer arithmetic, so that in principle our problem is solved. To make our algorithm faster, we will avoid computing the squares x^2 and y^2, by introducing the three new, nonnegative integer variables u, v and E denoting the differences between two successive squares and the 'error':

$$u = (x+1)^2 - x^2 = 2x + 1 \tag{4.9}$$

$$v = y^2 - (y-1)^2 = 2y - 1 \tag{4.10}$$

$$E = x^2 + y^2 - r^2 \tag{4.11}$$

Initially, we have $x = 0$ and $y = r$, so that $u = 1$, $v = 2r - 1$ and $E = 0$. In each step in the loop, we increase x by one, as previously discussed. Since, according to Eq. (4.9), this will increase the value of x^2 by u, we also have to increase E by u to satisfy Eq. (4.11). We can also see from Eq. (4.9) that increasing x by 1 implies that we have to increase u by 2. We now have to decide whether or not to decrease y by one. If we do, Eq. (4.10) indicates that the square y^2 decreases by v, so that according to Eq. (4.11) E also has to decrease by v. Since we want the absolute value of the error E to be as small as possible, the test we are looking for can be written

$$|E - v| < |E| \tag{4.12}$$

We will decrease y by 1 if and only if this test succeeds. It is interesting that we can write the condition (4.12) in a simpler form, by first replacing it with the equivalent test

$$(E - v)^2 < E^2$$

which can be simplified to

$$v(v - 2E) < 0$$

Since v is positive, we can simplify this further to

$$v < 2E$$

On the basis of the above discussion, we can now write the following method to draw the arc PQ (in which we write e instead of E):

```
void arc8(Graphics g, int r) {
   int x = 0, y = r, u = 1, v = 2 * r - 1, e = 0;
   while (x <= y) {
      putPixel(g, x, y);
      x++; e += u; u += 2;
      if (v < 2 * e){y--; e -= v; v -= 2;}
   }
}
```

Equations (4.10) and (4.11) show that in the case of decreasing y by 1, we have to decrease E by v and v by 2, as implemented in the if-statement. Note the symmetry between the three actions (related to y) in the if-statement and those (related to x) in the preceding program line.

The method *arc*8 is the basis for our final method, *drawCircle*, listed below. Besides drawing a full circle, it is also more general than *arc*8 in that it allows an arbitrary point C to be specified as the center of the circle. The comments in this method indicate directions of the compass. For example, *NNE* stands for north-northeast, which we use to refer to the arc between the north and the northeast directions (see Fig. 4.5). As usual, we think of the y-axis pointing upward, so that $y = r$ corresponds to north:

```
void drawCircle(Graphics g, int xC, int yC, int r) {
   int x = 0, y = r, u = 1, v = 2 * r - 1, e = 0;
   while (x < y) {
      putPixel(g, xC + x, yC + y); // NNE
      putPixel(g, xC + y, yC - x); // ESE
      putPixel(g, xC - x, yC - y); // SSW
      putPixel(g, xC - y, yC + x); // WNW
      x++; e += u; u += 2;
      if (v < 2 * e){y--; e -= v; v -= 2;}
```

```
      if (x > y) break;
      putPixel(g, xC + y, yC + x); // ENE
      putPixel(g, xC + x, yC - y); // SSE
      putPixel(g, xC - y, yC - x); // WSW
      putPixel(g, xC - x, yC + y); // NNW
   }
}
```

This version has been programmed in such a way that *putPixel* will not visit the same pixel more than once. This is important if we want to use the XOR paint mode, writing

```
g.setXORMode(Color.white);
```

before the call to *drawCircle*. Setting the paint mode in this way implies that black pixels are made white and vice versa, so that we can remove a circle in the same way as we draw it. It is then essential that a single call to *drawCircle* does not put the same pixel twice on the screen, for then it would be erased the second time. To prevent pixels from being used twice, the eight calls to *putPixel* in the loop are divided into two groups of four: those in the first group draw the arcs 0, 2, 4 and 6 (see Fig. 4.5), which include the starting points $(0, r)$, $(r, 0)$, $(0, -r)$ and $(-r, 0)$. The arcs 1, 3, 5 and 7 of the second group do not include these starting points because these calls to *putPixel* take place after x has been increased. As for the endpoint of each arc, we take care (a) that x is not greater than y for any call to *putPixel*, and (b) that the situation $x = y$ is covered by at most one of these two groups of *putPixel* calls. This situation will occur, for example, if $r = 6$; in this case the following eight points (relative to the center of the circle) are selected in this order, as far as the northeast quadrant of the circle (*NNE* and *ENE*) is concerned: (0, 6), (6, 1), (1, 6), (6, 2), (2, 6), (5, 3), (3, 5), (4, 4). In this situation the test after *while* terminates the loop. In contrast, the situation $x = y$ will not occur if $r = 8$, since in this case the 11 points (0, 8), (8, 1), (1, 8), (8, 2), (2, 8), (7, 3), (3, 7), (7, 4), (4, 7), (6, 5), (5, 6) are chosen, in that order. The loop now terminates because the break-statement in the middle of it is executed.

4.4 Cohen–Sutherland Line Clipping

In this section we will discuss how to draw line segments only as far as they lie within a given rectangle. For example, given a point P inside and a point Q outside a rectangle. It is then our task to draw only the line segment PI, where I is the point of intersection of PQ and the rectangle, as shown in Fig. 4.7. The rectangle is given by the four numbers x_{min}, x_{max}, y_{min} and y_{max}. These four values and the coordinates of P and Q are floating-point logical coordinates, as usual.

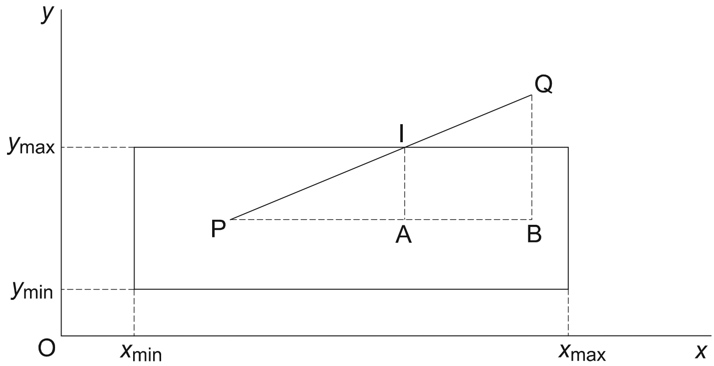


Fig. 4.7 Line segment PQ to be clipped

Since PQ intersects the upper edge of the rectangle in I, we have

$$y_I = y_{max}$$

As the triangles PAI and PBQ are similar, we can write

$$\frac{x_I - x_P}{y_I - y_P} = \frac{x_Q - x_P}{y_Q - y_P}$$

After replacing y_I with y_{max} and multiplying both sides of this equation by $y_{max} - y_P$, we easily obtain

$$x_I = x_P + \frac{(y_{max} - y_P)(x_Q - x_P)}{(y_Q - y_P)}$$

so that we can draw the desired line segment PI.

This easy way of computing the coordinates of point I is based on several facts that apply to Fig. 4.7, but that must not be relied on in a general algorithm. For example, if Q lies farther to the right, it may not be immediately clear whether the point of intersection lies on the upper edge $y = y_{max}$ or on the right edge $x = x_{max}$. In general, there are many more cases to consider. The logical decisions needed to find out which actions to take make line clipping an interesting topic from an algorithmic point of view. The Cohen–Sutherland algorithm solves this problem in an elegant and efficient way. We will express this algorithm in Java.

The four lines $x = x_{min}$, $x = x_{max}$, $y = y_{min}$, $y = y_{max}$ divide the xy-plane into nine regions. With any point P(x, y) we associate a four-bit code

$$b_3 b_2 b_1 b_0$$

identifying that region, as Fig. 4.8 shows.

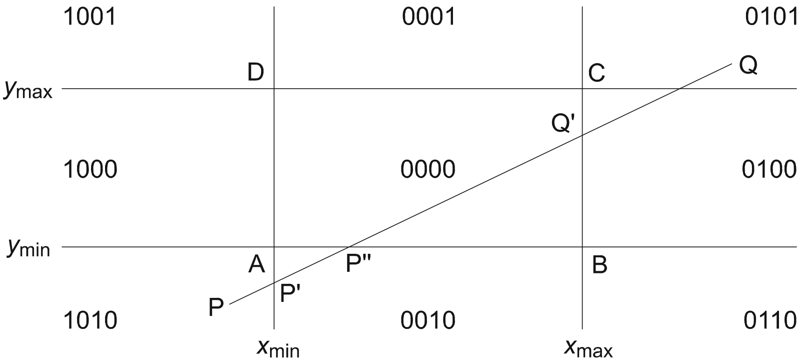


Fig. 4.8 Code values

For any point (x, y), the above four-bit code is defined as follows:

$$b_3 = 1 \text{ if and only if } x < x_{\min}$$
$$b_2 = 1 \text{ if and only if } x > x_{\max}$$
$$b_1 = 1 \text{ if and only if } y < y_{\min}$$
$$b_0 = 1 \text{ if and only if } y > y_{\max}$$

Based on this code, the Cohen–Sutherland algorithm replaces the endpoints P and Q of a line segment, if they lie outside the rectangle, with points of intersection of PQ and the rectangle, that is, if there are such points of intersection. This is done in a few steps. For example, in Fig. 4.8, the following steps are taken:

1. Since P lies to the left of the left rectangle edge, it is replaced with P′ (on $x = x_{\min}$), so that only P′Q remains to be dealt with.
2. Since P′ lies below the lower rectangle edge, it is replaced with P″ (on $y = y_{\min}$), so that P″Q remains to be dealt with.
3. Since Q lies to the right of the right rectangle edge, it is replaced with Q′ (on $x = x_{\max}$), so that P″Q′ remains to be dealt with.
4. Line segment P″Q′ is drawn.

The steps 1, 2 and 3 are done in a loop, which can terminate in two ways:

- If the four-bit codes of P (or P′ or P″, which we again refer to as P in the program) and of Q are equal to zero; the (new) line segment PQ is then drawn.
- If the two four-bit codes contain a 1-bit in the same position; this implies that P and Q are on the same side of a rectangle edge, so that nothing has to be drawn.

The method *clipLine* in the following program shows this in greater detail:

```
// ClipLine.java: Cohen-Sutherland line clipping.
import java.awt.*;
import java.awt.event.*;

public class ClipLine extends Frame {
   public static void main(String[] args) {new ClipLine();}

   ClipLine() {
      super("Click on two opposite corners of a rectangle");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(500, 300);
      add("Center", new CvClipLine());
      setCursor(Cursor.getPredefinedCursor(Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvClipLine extends Canvas {
   float xmin, xmax, ymin, ymax, rWidth = 10.0F, rHeight = 7.5F,
         pixelSize;
   int maxX, maxY, centerX, centerY, np = 0;

   CvClipLine() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            float x = fx(evt.getX()), y = fy(evt.getY());
            if (np == 2) np = 0;
            if (np == 0) {xmin = x; ymin = y;
            }
            else {
               xmax = x; ymax = y;
               if (xmax < xmin) {
                  float t = xmax; xmax = xmin; xmin = t;
               }
               if (ymax < ymin) {
                  float t = ymax; ymax = ymin; ymin = t;
               }
            }
            np++;
            repaint();
         }
      });
   }
```

```
void initgr() {
   Dimension d = getSize();
   maxX = d.width - 1; maxY = d.height - 1;
   pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
   centerX = maxX / 2; centerY = maxY / 2;
}

int iX(float x) {return Math.round(centerX + x / pixelSize);}
int iY(float y) {return Math.round(centerY - y / pixelSize);}
float fx(int x) {return (x - centerX) * pixelSize;}
float fy(int y) {return (centerY - y) * pixelSize;}

void drawLine(Graphics g, float xP, float yP, float xQ, float yQ) {
   g.drawLine(iX(xP), iY(yP), iX(xQ), iY(yQ));
}

int clipCode(float x, float y) {
   return (x < xmin ? 8 : 0) | (x > xmax ? 4 : 0) |
          (y < ymin ? 2 : 0) | (y > ymax ? 1 : 0);
}

void clipLine(Graphics g, float xP, float yP, float xQ, float yQ,
      float xmin, float ymin, float xmax, float ymax) {
   int cP = clipCode(xP, yP), cQ = clipCode(xQ, yQ);
   float dx, dy;
   while ((cP | cQ) != 0) {
      if ((cP & cQ) != 0) return;
      dx = xQ - xP; dy = yQ - yP;
      if (cP != 0) {
         if ((cP & 8) == 8) {yP += (xmin - xP) * dy / dx; xP = xmin;}
         else
         if ((cP & 4) == 4) {yP += (xmax - xP) * dy / dx; xP = xmax;}
         else
         if ((cP & 2) == 2) {xP += (ymin - yP) * dx / dy; yP = ymin;}
         else
         if ((cP & 1) == 1) {xP += (ymax - yP) * dx / dy; yP = ymax;}
         cP = clipCode(xP, yP);
      }
      else if (cQ != 0) {
         if ((cQ & 8) == 8) {yQ += (xmin - xQ) * dy / dx; xQ = xmin;}
         else
         if ((cQ & 4) == 4) {yQ += (xmax - xQ) * dy / dx; xQ = xmax;}
         else
         if ((cQ & 2) == 2) {xQ += (ymin - yQ) * dx / dy; yQ = ymin;}
         else
```

```
            if ((cQ & 1) == 1) {xQ += (ymax - yQ) * dx / dy; yQ = ymax;}
            cQ = clipCode(xQ, yQ);
         }
      }
      drawLine(g, xP, yP, xQ, yQ);
   }

   public void paint(Graphics g) {
      initgr();
      if (np == 1) { // Draw horizontal and vertical lines through
                     // first defined point:
         drawLine(g, fx(0), ymin, fx(maxX), ymin);
         drawLine(g, xmin, fy(0), xmin, fy(maxY));
      }
      else
      if (np == 2) { // Draw rectangle:
         drawLine(g, xmin, ymin, xmax, ymin);
         drawLine(g, xmax, ymin, xmax, ymax);
         drawLine(g, xmax, ymax, xmin, ymax);
         drawLine(g, xmin, ymax, xmin, ymin);

         // Draw 20 concentric regular pentagons, as
         // far as they lie within the rectangle:
         float rMax = Math.min(rWidth, rHeight) / 2,
              deltaR = rMax / 20, dPhi = (float) (0.4 * Math.PI);

         for (int j = 1; j <= 20; j++) {
            float r = j * deltaR;
            // Draw a pentagon:
            float xA, yA, xB = r, yB = 0;

            for (int i = 1; i <= 5; i++) {
               float phi = i * dPhi;
               xA = xB; yA = yB;
               xB = (float) (r * Math.cos(phi));
               yB = (float) (r * Math.sin(phi));
               clipLine(g, xA, yA, xB, yB, xmin, ymin, xmax, ymax);
            }
         }
      }
   }
}
```

The program draws 20 concentric (regular) pentagons, as far as these lie within a rectangle, which the user can define by clicking on any two opposite corners.

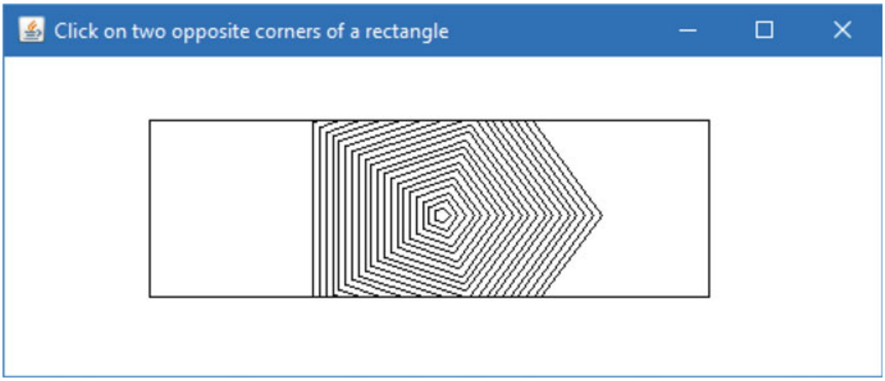


Fig. 4.9 Demonstration of program *ClipLine.java*

When he or she clicks for the third time, the situation is the same as at the beginning: the screen is cleared and a new rectangle can be defined, in which again parts of 20 pentagons appear, and so on. As usual, if the user changes the window dimensions, the size of the result is changed accordingly. Figure 4.9 shows the situation just after the pentagons are drawn.

If we look at the while-loop in the method *clipLine*, it seems that this code is somewhat inefficient because of the divisions dy/dx and dx/dy inside that loop while dx and dy are not changed in it. However, we should bear in mind that dx or dy may be zero. The if-statements in the loop guarantee that no division by dx or dy will be performed if that variable is zero. Besides, this loop is different from most other program loops in that the inner part is usually executed only once or not at all, and rarely more than once.

4.5 Sutherland–Hodgman Polygon Clipping

In contrast to line clipping, discussed in the previous section, we will now deal with polygon clipping, which is different in that it converts a polygon into another one within a given rectangle, as Figs. 4.10 and 4.11 illustrate.

The program that we will discuss draws a fixed rectangle and enables the user to specify the vertices of a polygon by clicking, in the same way as discussed in Sect. 1.4. As long as the first vertex, in Fig. 4.10 on the left, is not selected for the second time, successive vertices are connected by polygon edges. As soon as the first vertex is selected again, the polygon is clipped, as Fig. 4.11 shows. Some vertices of the original polygon do not belong to the clipped one. On the other hand, the latter polygon has some new vertices, which are all points of intersection of the edges of the original polygon and those of the rectangle. In general, the number of vertices of the clipped polygon can be greater than, equal to or less than that of the original one. In Fig. 4.11 there are five new polygon edges, which are part of the rectangle edges.

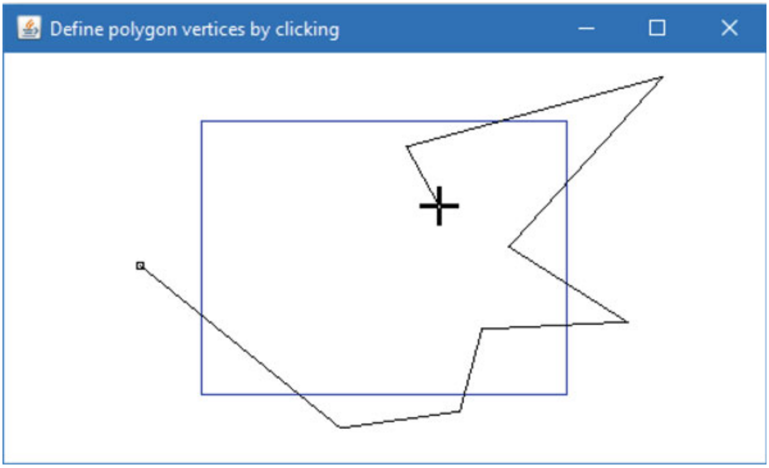


Fig. 4.10 Nine polygon vertices defined; final edge not yet drawn

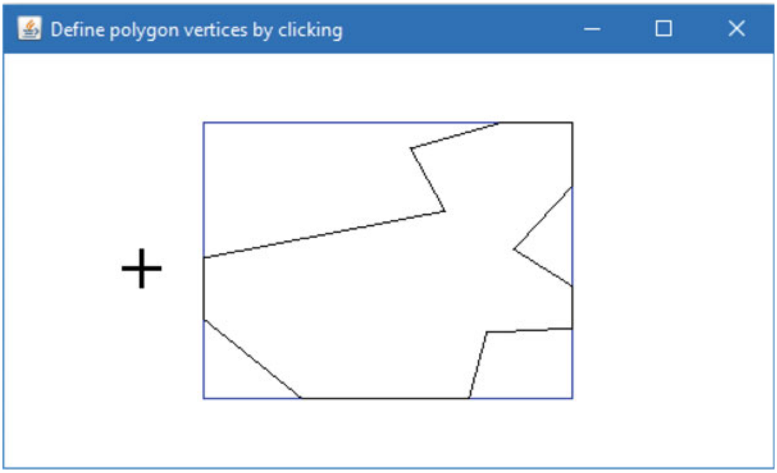


Fig. 4.11 Polygon completed and clipped

The program that produced Fig. 4.11 is based on the Sutherland–Hodgman algorithm, which first clips all polygon edges against one rectangle edge, or rather, the infinite line through such an edge. This results in a new polygon, which is then clipped against the next rectangle edge, and so on. Figure 4.12 shows a rectangle and a polygon, ABCDEF. Starting in vertex A, we find that AB intersects the line $x = x_{max}$ in point I, which will be a new vertex of the clipped polygon. The same applies to point J, the point of intersection of DE with the same vertical rectangle edge.

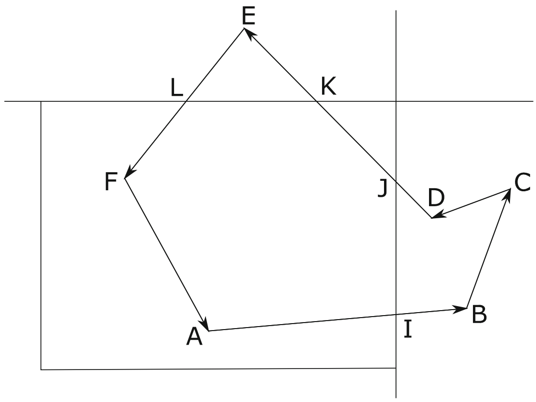


Fig. 4.12 The Sutherland–Hodgman algorithm

Let us refer to the original polygon as the *input polygon* and the new one as the *output polygon*. We represent each polygon by a sequence of successive vertices. Let us start with the right rectangle edge and treat it as an infinite clipping line, that can be expressed as $x = x_{max}$, and see how to clip the polygon against this line. In general, when working on one clipping line, we ignore all other rectangle edges. Initially, the output polygon is empty. When following all successive polygon edges such as AB, we focus on the endpoint, B, and decide as follows which points will belong to the output polygon:

> If A and B lie on different sides of the clipping line, the point of intersection, I, is added to the output polygon. Regardless of this being the case, B is added to the output polygon if and only if it lies on the same side of the clipping line as the rectangle.

Starting with the directed edge AB in Fig. 4.12, point I is the first to be added to the output polygon. Vertex B is not added because it is not on the same side of the clipping line as the rectangle, and the same applies to the endpoints of the following directed edges, BC and CD. When dealing with edge DE, we first add J and then E to the output polygon since they are both on the same side of the clipping line as the rectangle. The endpoints of the next two edges, EF and FA lie on the same side of the clipping line as the rectangle and are therefore added to the output polygon. In this way we obtain the vertices I, J, E, F, and A, in that order. We then use this output polygon as the input polygon to clip against the top rectangle edge. Using the same method as described above on the right edge, we obtain the vertices I, J, K, L, F, and A, which forms the output polygon IJKLFA. Although in this example IJKLFA is the desired clipped polygon, it will in general be necessary to use the output polygon as input for working with another rectangle edge. The program below shows an implementation in Java:

```
// ClipPoly.java: Clipping a polygon.
// Uses: Point2D (Section 1.4).
import java.awt.*;
import java.awt.event.*;
import java.util.*;
```

```
public class ClipPoly extends Frame {
   public static void main(String[] args) {new ClipPoly();   }

   ClipPoly() {
      super("Define polygon vertices by clicking");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(500, 300);
      add("Center", new CvClipPoly());
      setCursor(Cursor.getPredefinedCursor(Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvClipPoly extends Canvas {
   Poly poly = null;
   float rWidth = 10.0F, rHeight = 7.5F, pixelSize;
   int x0, y0, centerX, centerY;
   boolean ready = true;

   CvClipPoly() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            int x = evt.getX(), y = evt.getY();
            if (ready) {
               poly = new Poly();
               x0 = x; y0 = y;
               ready = false;
            }
            if (poly.size() > 0 &&
                  Math.abs(x - x0) < 3 && Math.abs(y - y0) < 3)
               ready = true;
            else
               poly.addVertex(new Point2D(fx(x), fy(y)));
            repaint();
         }
      });
   }

   void initgr() {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
   }
```

```
int iX(float x) {return Math.round(centerX + x / pixelSize);}
int iY(float y) {return Math.round(centerY - y / pixelSize);}
float fx(int x) {return (x - centerX) * pixelSize;}
float fy(int y) {return (centerY - y) * pixelSize;}

void drawLine(Graphics g, float xP, float yP, float xQ, float yQ) {
   g.drawLine(iX(xP), iY(yP), iX(xQ), iY(yQ));
}

void drawPoly(Graphics g, Poly poly) {
   int n = poly.size();
   if (n == 0) return;
   Point2D a = poly.vertexAt(n - 1);
   for (int i = 0; i < n; i++) {
      Point2D b = poly.vertexAt(i);
      drawLine(g, a.x, a.y, b.x, b.y);
      a = b;
   }
}

public void paint(Graphics g) {
   initgr();
   float xmin = -rWidth / 3, xmax = rWidth / 3,
         ymin = -rHeight / 3, ymax = rHeight / 3;
   // Draw clipping rectangle:
   g.setColor(Color.blue);
   drawLine(g, xmin, ymin, xmax, ymin);
   drawLine(g, xmax, ymin, xmax, ymax);
   drawLine(g, xmax, ymax, xmin, ymax);
   drawLine(g, xmin, ymax, xmin, ymin);
   g.setColor(Color.black);
   if (poly == null) return;
   int n = poly.size();
   if (n == 0) return;
   Point2D a = poly.vertexAt(0);
   if (!ready) { // Show tiny rectangle around first vertex:
      g.drawRect(iX(a.x) - 2, iY(a.y) - 2, 4, 4);
      // Draw incomplete polygon:
      for (int i = 1; i < n; i++) {
         Point2D b = poly.vertexAt(i);
         drawLine(g, a.x, a.y, b.x, b.y);
         a = b;
      }
   }
```

```
      else {
         poly.clip(xmin, ymin, xmax, ymax);
         drawPoly(g, poly);
      }
   }
}

class Poly {
   Vector<Point2D> v = new Vector<Point2D>();
   void addVertex(Point2D p) {v.addElement(p);}
   int size() {return v.size();}

   Point2D vertexAt(int i) {
      return (Point2D) v.elementAt(i);
   }

   void clip(float xmin, float ymin, float xmax, float ymax) {
      // Sutherland-Hodgman polygon clipping:
      Poly poly1 = new Poly();
      int n;
      Point2D a, b;
      boolean aIns, bIns;
      // Tells whether A or B is on the same side as the rectangle

      // Clip against x == xmax:
      if ((n = size()) == 0) return;
      b = vertexAt(n - 1);
      for (int i = 0; i < n; i++) {
         a = b; b = vertexAt(i);
         aIns = a.x <= xmax; bIns = b.x <= xmax;
         if (aIns != bIns)
            poly1.addVertex(new Point2D(xmax,
                 a.y + (b.y - a.y) * (xmax - a.x) / (b.x - a.x)));
         if (bIns) poly1.addVertex(b);
      }
      v = poly1.v; poly1 = new Poly();

      // Clip against x == xmin:
      if ((n = size()) == 0) return;
      b = vertexAt(n - 1);
      for (int i = 0; i < n; i++) {
         a = b; b = vertexAt(i);
         aIns = a.x >= xmin; bIns = b.x >= xmin;
```

```
         if (aIns != bIns)
            poly1.addVertex(new Point2D(xmin,
                  a.y + (b.y - a.y) * (xmin - a.x) / (b.x - a.x)));
         if (bIns) poly1.addVertex(b);
      }
      v = poly1.v; poly1 = new Poly();

      // Clip against y == ymax:
      if ((n = size()) == 0) return;
      b = vertexAt(n - 1);
      for (int i = 0; i < n; i++) {
         a = b; b = vertexAt(i);
         aIns = a.y <= ymax; bIns = b.y <= ymax;
         if (aIns != bIns)
            poly1.addVertex(new Point2D(a.x +
                  (b.x - a.x) * (ymax - a.y) / (b.y - a.y), ymax));
         if (bIns) poly1.addVertex(b);
      }
      v = poly1.v; poly1 = new Poly();

      // Clip against y == ymin:
      if ((n = size()) == 0) return;
      b = vertexAt(n - 1);
      for (int i = 0; i < n; i++) {
         a = b; b = vertexAt(i);
         aIns = a.y >= ymin; bIns = b.y >= ymin;
         if (aIns != bIns)
            poly1.addVertex(new Point2D(a.x +
                  (b.x - a.x) * (ymin - a.y) / (b.y - a.y), ymin));
         if (bIns) poly1.addVertex(b);
      }
      v = poly1.v; poly1 = new Poly();
   }
}
```

The Sutherland–Hodgman algorithm can be adapted for clipping regions other than rectangles and for three-dimensional applications.

4.6 Bézier Curves

There are many algorithms for constructing curves. A particularly elegant and practical one is based on specifying four points that completely determine a curve segment: two endpoints and two control points. Curves constructed in this way are referred to as (cubic) Bézier curves. In Fig. 4.13, we have the endpoints P_0 and P_3,

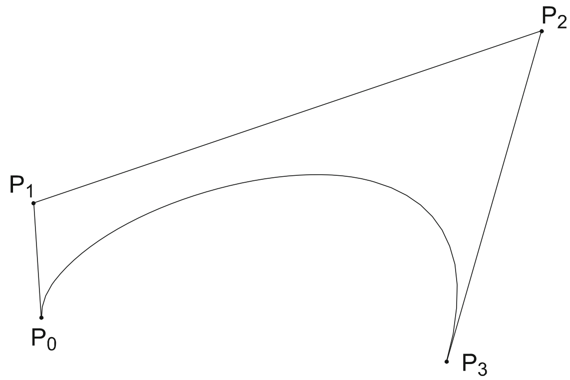


Fig. 4.13 Bézier curve based on four points

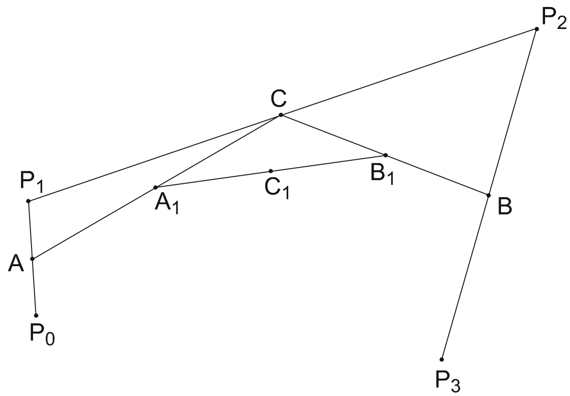


Fig. 4.14 Constructing points for two smaller curve segments

the control points P_1 and P_2, and the curve constructed on the basis of these four points.

Writing a method to draw this curve is surprisingly easy, provided we use recursion. As Fig. 4.14 shows, we compute six midpoints, namely:

- A, the midpoint of P_0P_1
- B, the midpoint of P_2P_3
- C, the midpoint of P_1P_2
- A_1, the midpoint of AC
- B_1, the midpoint of BC
- C_1, the midpoint of A_1B_1

After this, we can divide the original task of drawing the Bézier curve P_0P_3 (with control points P_1 and P_2) into two simpler tasks:

- drawing the Bézier curve P_0C_1, with control points A and A_1
- drawing the Bézier curve C_1P_3, with control points B_1 and B

The two tasks need to be done only if the original points P_0 and P_3 are further apart than some small distance, say, ε. Otherwise, we simply draw the straight line P_0P_3. Since we are using pixels on a raster, we can also base the test just mentioned on device coordinates: we will simply draw a straight line from P_0 to P_3 if and only if the corresponding pixels are neighbors or identical, writing.

```
if (Math.abs(x0 - x3) <= 1 && Math.abs(y0 - y3) <= 1)
      g.drawLine(x0, y0, x3, y3);
else ...
```

The recursive method *bezier* in the following program shows an implementation of this algorithm. The program expects the user to specify the four points P_0, P_1, P_2 and P_3, in that order, by clicking the mouse. After the fourth point, P_3, has been specified, the curve is drawn. Any new mouse clicking is interpreted as the first point, P_0, of a new curve; the previous curve simply disappears and another curve can be constructed in the same way as the first one, and so on.

```
// Bezier.java: Bezier curve segments.
// Uses: Point2D (Section 1.4).
import java.awt.*;
import java.awt.event.*;

public class Bezier extends Frame {
   public static void main(String[] args) {new Bezier();}

   Bezier() {
      super("Define endpoints and control points of curve segment");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(500, 300);
      add("Center", new CvBezier());
      setCursor(Cursor.getPredefinedCursor(Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvBezier extends Canvas {
   Point2D[] p = new Point2D[4];
   int np = 0, centerX, centerY;
   float rWidth = 10.0F, rHeight = 7.5F, eps = rWidth / 100F,
         pixelSize;
```

```
CvBezier() {
   addMouseListener(new MouseAdapter() {
      public void mousePressed(MouseEvent evt) {
         float x = fx(evt.getX()), y = fy(evt.getY());
         if (np == 4) np = 0;
         p[np++] = new Point2D(x, y);
         repaint();
      }
   });
}

void initgr() {
   Dimension d = getSize();
   int maxX = d.width - 1, maxY = d.height - 1;
   pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
   centerX = maxX / 2; centerY = maxY / 2;
}

int iX(float x) {return Math.round(centerX + x / pixelSize);}
int iY(float y) {return Math.round(centerY - y / pixelSize);}
float fx(int x) {return (x - centerX) * pixelSize;}
float fy(int y) {return (centerY - y) * pixelSize;}

Point2D middle(Point2D a, Point2D b) {
   return new Point2D((a.x + b.x) / 2, (a.y + b.y) / 2);
}

void bezier(Graphics g, Point2D p0, Point2D p1, Point2D p2,
      Point2D p3) {
   int x0 = iX(p0.x), y0 = iY(p0.y),
       x3 = iX(p3.x), y3 = iY(p3.y);
   if (Math.abs(x0 - x3) <= 1 && Math.abs(y0 - y3) <= 1)
       g.drawLine(x0, y0, x3, y3);
   else {
      Point2D
         a = middle(p0, p1), b = middle(p3, p2), c = middle(p1, p2),
         a1 = middle(a, c), b1 = middle(b, c), c1 = middle(a1, b1);
      bezier(g, p0, a, a1, c1);
      bezier(g, c1, b1, b, p3);
   }
}

public void paint(Graphics g) {
   initgr();
   int left = iX(-rWidth / 2), right = iX(rWidth / 2),
       bottom = iY(-rHeight / 2), top = iY(rHeight / 2);
   g.drawRect(left, top, right - left, bottom - top);
```

```
      for (int i = 0; i < np; i++) {
         // Show tiny rectangle around point:
         g.drawRect(iX(p[i].x) - 2, iY(p[i].y) - 2, 4, 4);
         if (i > 0)
            // Draw line p[i-1]p[i]:
            g.drawLine(iX(p[i - 1].x), iY(p[i - 1].y),
                       iX(p[i].x), iY(p[i].y));
      }
      if (np == 4) bezier(g, p[0], p[1], p[2], p[3]);
   }
}
```

Since this program uses the isotropic mapping mode with a logical coordinate ranges 0–10.0 for x and 0–7.5 for y, we should only use a rectangle whose height is 75% of its width. As in Sects. 1.3 and 1.4, we place this rectangle in the center of the screen and make it as large as possible. It is shown in Fig. 4.15; if the four points for the curve are chosen within this rectangle, they will be visible regardless of how the size of the window is changed by the user. The same applies to the curve, which is automatically scaled, in the same way as we did in Sects. 1.3 and 1.4.

This way of constructing a Bézier curve may look like magic. To understand what is going on, we must be familiar with the notion of *parametric representation*, which, in 2D, we can write as

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

where t is a parameter, which we may think of as *time*. Variable t ranges from 0 to 1: as if we move from P_0 to P_3 with constant velocity, starting at $t = 0$ and finishing at

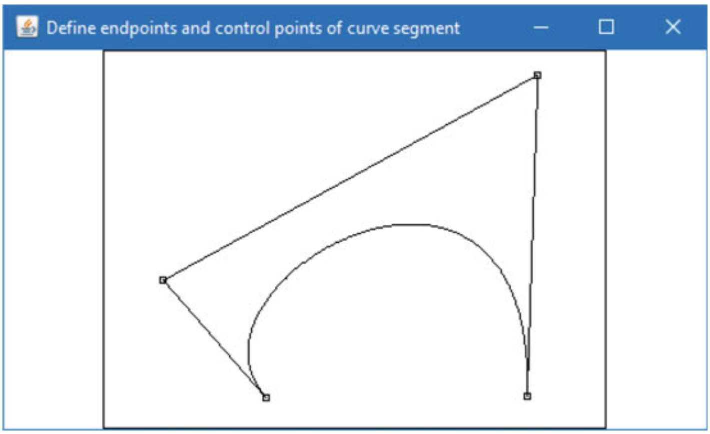


Fig. 4.15 A constructed Bézier curve

$t = 1$. At time $t = 0.5$ we are half-way. With cubic Bézier curves, both $f(t)$ and $g(t)$ are 3rd degree polynomials in t.

Before we proceed, let us pay some attention to expressions such as

$$\mathrm{A} = \frac{1}{2}(\mathrm{P}_0 + \mathrm{P}_1)$$

to compute the midpoint A of line segment $\mathrm{P}_0\mathrm{P}_1$. The points in such expressions actually denote vectors, which enables us to form their sum and to multiply them by a scalar. Without this shorthand notation, we would have to write

$$\mathbf{OA} = \frac{1}{2}(\mathbf{OP_0} + \mathbf{OP_1})$$

which would be rather awkward. Let us write each P_i as the column vector

$$\mathrm{P}_i = \begin{pmatrix} P[i].x \\ P[i].y \end{pmatrix}$$

(where we have taken the liberty of mixing mathematical and programming notations). In the same way, we combine the above functions $f(t)$ and $g(t)$, writing

$$B(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

Then the cubic Bézier curve is defined as follows:

$$B(t) = (1-t)^3\mathrm{P}_0 + 3t(1-t)^2\mathrm{P}_1 + 3t^2(1-t)\mathrm{P}_2 + t^3\mathrm{P}_3 \tag{4.13}$$

Substituting 0 and 1 for t, we find $B(0) = \mathrm{P}_0$ and $B(1) = \mathrm{P}_3$. Before discussing the relation between this definition and the recursive midpoint construction, let us see this definition in action. It enables us to replace the recursive method *bezier* with the following non-recursive one:

```
void bezier1(Graphics g, Point2D[] p) {
   int n = 200;
   float dt = 1.0F/n, x = p[0].x, y = p[0].y, x0, y0;
   for (int i=1; i<=n; i++) {
      float t = i * dt, u = 1 - t,
            tuTriple = 3 * t * u,
            c0 = u * u * u,
            c1 = tuTriple * u,
            c2 = tuTriple * t,
            c3 = t * t * t;
```

```
      x0 = x; y0 = y;
      x = c0*p[0].x + c1*p[1].x + c2*p[2].x + c3*p[3].x;
      y = c0*p[0].y + c1*p[1].y + c2*p[2].y + c3*p[3].y;
      g.drawLine(iX(x0), iY(y0), iX(x), iY(y));
   }
}
```

This method produces the same curve as that by *bezier*, provided we also replace the call to *bezier* with this one:

```
bezier1(g, P);
```

We will discuss a more efficient non-recursive method, *bezier*2, equivalent to *bezier*1, at the end of this section.

Since $B(t)$ denotes the position at time t, the derivative $B'(t)$ of this function (which is also a column vector depending on t) can be regarded as the *velocity*. After some algebraic manipulation and differentiating, we find

$$B'(t) = -3(t-1)^2\mathrm{P}_0 + 3(3t-1)(t-1)\mathrm{P}_1 - 3t(3t-2)\mathrm{P}_2 + 3t^2\mathrm{P}_3 \qquad (4.14)$$

which gives

$$B'(0) = 3(\mathrm{P}_1 - \mathrm{P}_0)$$
$$B'(1) = 3(\mathrm{P}_3 - \mathrm{P}_2)$$

These two results are velocity vectors at the starting point P_0 and the endpoint P_3. They show that the direction in which the curve is drawn along the vectors $\mathrm{P}_0\mathrm{P}_1$ and $\mathrm{P}_2\mathrm{P}_3$, as Fig. 4.16 illustrates.

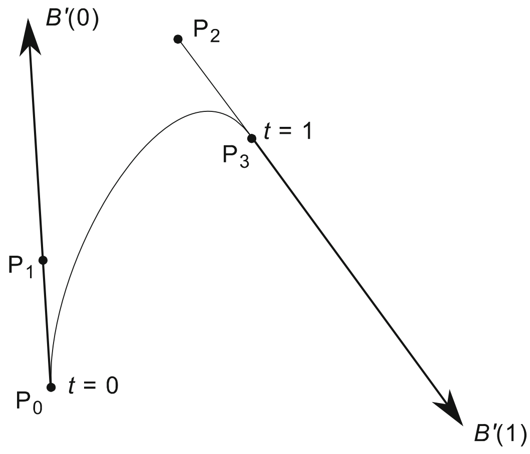


Fig. 4.16 Velocity at the points P_0 and P_3

We have been discussing two entirely different ways to construct a curve between the points P_0 and P_3, and, without an experiment, it is not clear that these curves are identical. For the time being, we will distinguish between the two curves and refer to them as

- the *midpoint curve*, constructed by a recursive process of computing midpoints, and implemented in the method *bezier*;
- the *analytical curve*, given by Eq. (4.13) and implemented in the method *bezier*1.

Although both methods are based on the four points P_0, P_1, P_2 and P_3, the ways we compute hundreds of curve points (to connect them by tiny straight lines) are very different. It would be unsatisfactory if it remained a mystery why these curves are identical. Let us therefore briefly discuss a way to prove this fact.

Using the function $B(t)$ of Eq. (4.13), we find

$$B(0.5) = \frac{1}{8}\,(P_0 + 3P_1 + 3P_3 + P_3)$$

Since point C_1 in Fig. 4.14 was used to divide the whole midpoint curve into two smaller curves, we might suspect that C_1 and $B(0.5)$ are two different expressions for the same point. We verify this by expressing C_1 in terms of the four given points, using Fig. 4.14.

Because C_1 is the midpoint of A_1B_1 and both A_1 and B_1 are also midpoints, and so on, we find

$$\begin{aligned} A &= \frac{1}{2}(P_0 + P_1) \\ B &= \frac{1}{2}(P_2 + P_3) \\ C &= \frac{1}{2}(P_1 + P_2) \end{aligned} \tag{4.15}$$

$$\begin{aligned} A_1 &= \frac{1}{2}(A + C) = \frac{1}{4}(P_0 + 2P_1 + P_2) \\ B_1 &= \frac{1}{2}(C + B) = \frac{1}{4}(P_1 + 2P_2 + P_3) \end{aligned} \tag{4.16}$$

$$C_1 = \frac{1}{2}\,(A_1 + B_1) = \frac{1}{8}\,(P_0 + 3P_1 + 3P_2 + P_3) \tag{4.17}$$

which is indeed the expression that we also had found for $B(0.5)$. This proves that point C_1, which obviously belongs to the midpoint curve, also lies on the analytical curve. Besides P_0 and P_3, there is now only one point, C_1, of which we have proved that it lies on both curves, so it seems we are still far away from the proof that these curves are identical. However, we can now apply the same argument recursively, which would enable us to find as many points that lie on both curves as we like. Restricting ourselves to the first half of each curve, we focus on the points P_0, A, A_1 and C_1, which we can again use to construct both midpoint and analytical curves. For the latter we use the following equation, which is similar to Eq. (4.13):

$$b(u) = (1-u)^3 P_0 + 3u(1-u)^2 A + 3u^2(1-u)A_1 + u^3 C_1 \tag{4.18}$$

It is then obvious that $b(0) = P_0$, $b(1) = C_1$ and $b(0.5)$ is identical with a midpoint (between P_0 and C_1) used in the recursive process, in the same way as $B(0.5)$ being identical with the midpoint C_1. There is one remaining difficulty: is the analytical curve given by Eq. (4.18) really the same as the first part of that given by Eq. (4.13)? We will show that this is indeed the case. Using $u = 2\,t$ (since $u = 1$ and $t = 0.5$ at point C_1) we have

$$b(u) = B(t)$$

To verify this, note that, according to Eq. (4.18), we have

$$b(u) = b(2t) = (1-2t)^3 P_0 + 6t(1-2t)^2 A + 12t^2(1-2t)A_1 + 8t^3 C_1$$

Using Eqs. (4.15), (4.16) and (4.17), we can write the last expression as

$$\begin{aligned}&(1-2t)^3 P_0 + 3t(1-2t)^2(P_0 + P_1) + 3t^2(1-2t)(P_0 + 2P_1 + P_2)\\ &\quad + t^3(P_0 + 3P_1 + 3P_2 + P_3)\end{aligned}$$

Rearranging this formula, we find that it is equal to the expression for $B(t)$ in Eq. (4.13), which is what we had to prove.

Building Smooth Curves from Curve Segments

Suppose that we want to combine two Bézier curve segments, one based on the four points P_0, P_1, P_2, P_3 and the other on Q_0, Q_1, Q_2 and Q_3, in such a way that the endpoint P_3 of the first segment coincides with the starting point Q_0 of the second. Then the combined curve will be smoothest if the final velocity $B'(1)$ (see Fig. 4.16) of the first segment is equal to the initial velocity $B'(0)$ of the second. This will be the case if the point P_3 ($= Q_0$) lies exactly in the middle of the line segment P_2Q_1. The high degree of smoothness obtained in this way is referred to as second-order continuity. It implies that not only the two segments have the same tangent in their common point $P_3 = Q_0$, but also the curvature is continuous in this point. By contrast, we have first-order continuity if P_3 lies on the line segment P_2Q_1 but not in the middle of it. In this case, although the curve looks reasonably smooth because both segments have the same tangent in the common point $P_3 = Q_0$, there is a discontinuity in the curvature in this point.

Matrix Notation

It will be clear that we can write Eq. (4.13) as follows:

$$B(t) = \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \begin{bmatrix} \mathrm{P}_0 \\ \mathrm{P}_1 \\ \mathrm{P}_2 \\ \mathrm{P}_3 \end{bmatrix} \tag{4.19}$$

Since the row vector in this matrix product is equal to

$$\begin{bmatrix} -t^3+3t^2-3t+1 & 3t^3-6t^2+3t & -3t^3+3t^2 & t^3 \end{bmatrix}$$

we can also write it as the product of a simpler row vector, $[t^3 \quad t^2 \quad t \quad 1]$, and a 4×4 matrix, obtaining the following result for the Bézier curve:

$$B(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{P}_0 \\ \mathrm{P}_1 \\ \mathrm{P}_2 \\ \mathrm{P}_3 \end{bmatrix} \tag{4.20}$$

As we know, any matrix product ABC of three matrices is equal to both $(AB)C$ and $A(BC)$. If we do the first matrix multiplication first, as in $(AB)C$, Eq. (4.20) reduces to Eq. (4.19). On the other hand, if we do the second first, as in $A(BC)$, we obtain the following result:

$$B(t) = (-\mathrm{P}_0 + 3\mathrm{P}_1 - 3\mathrm{P}_2 + \mathrm{P}_3)t^3 + 3(\mathrm{P}_0 - 2\mathrm{P}_1 + \mathrm{P}_3)t_2 - 3(\mathrm{P}_1 - \mathrm{P}_0)t + \mathrm{P}_0$$

This is interesting because it provides us with a very efficient way of drawing a Bézier curve segment, as the following improved method shows:

```
void bezier2(Graphics g, Point2D[] p) {
   int n = 200;
   float dt = 1.0F/n,
         cx3 = -p[0].x + 3 * (p[1].x - p[2].x) + p[3].x,
         cy3 = -p[0].y + 3 * (p[1].y - p[2].y) + p[3].y,
         cx2 = 3 * (p[0].x - 2 * p[1].x + p[2].x),
         cy2 = 3 * (p[0].y - 2 * p[1].y + p[2].y),
         cx1 = 3 * (p[1].x - p[0].x),
         cy1 = 3 * (p[1].y - p[0].y),
         cx0 = p[0].x,
         cy0 = p[0].y,
         x = p[0].x, y = p[0].y, x0, y0;
```

```
    for (int i=1; i<=n; i++) {
       float t = i * dt;
       x0 = x; y0 = y;
       x = ((cx3 * t + cx2) * t + cx1) * t + cx0;
       y = ((cy3 * t + cy2) * t + cy1) * t + cy0;
       g.drawLine(iX(x0), iY(y0), iX(x), iY(y));
    }
}
```

The above computation of x and y is an application of *Horner's rule*, according to which we can efficiently compute polynomials by using the right-hand rather than the left-hand side of the following equation:

$$a_3t^3 + a_2t^2 + a_1t + a_0 = ((a_3t + a_2)t + a_1)t + a_0$$

Although *bezier*2 does not look simpler than *bezier*1, it is much more efficient because of the reduced number of arithmetic operations in the for-loop. With a large number of steps, such as $n = 200$ in these versions *bezier*1 and *bezier*2, it is the number of operations inside the loop that counts, not the preparatory actions that precede the loop.

3D Curves

Although the curves discussed here are two-dimensional, three-dimensional curves can be generated in the same way. We simply add a z-component to $B(t)$ and to the control points, and compute this in the same way as the x- and y-components are computed. The possibility of generating curves that do not lie in a plane is related to the degree 3 of the polynomials we have been discussing. If the four given points do not lie in the same plane, the generated cubic curve segment does not either. By contrast, quadratic curves are determined by only three points, which uniquely define a plane (unless they are collinear); the quadratic curve through those three points lies in that plane. In other words, polynomial curves can be non-planar only if they are at least of degree 3.

4.7 B-Spline Curve Fitting

Besides the techniques discussed in the previous section, there are other ways of generating curves $x = f(t)$, $y = g(t)$, where f and g are polynomials in t of degree 3. A popular one, known as *B-splines*, has the characteristic that the generated curve will normally not pass through the given points. We will refer to all these points as *control points*. A single segment of such a curve, based on four control points A, B,

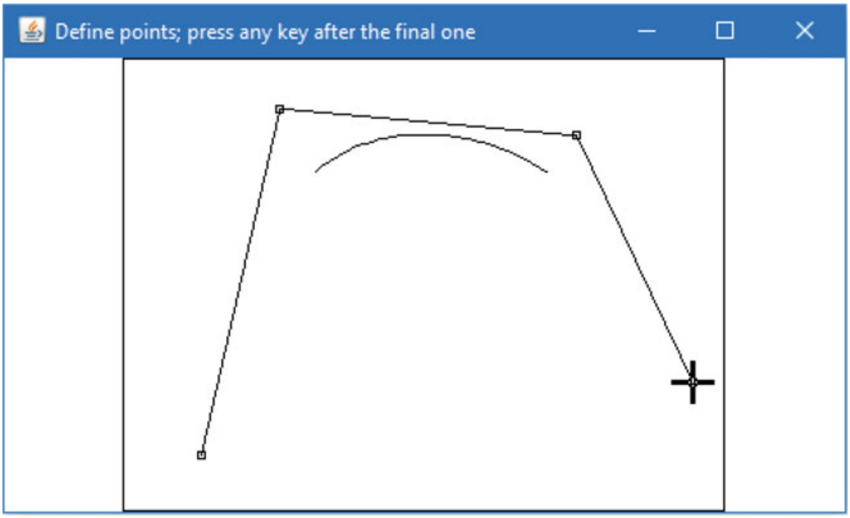


Fig. 4.17 Single B-spline segment, based on four points

C and D looks rather disappointing in that it seems to be related only to B and C. This is shown in Fig. 4.17, in which, from left to right, the points A, B, C and D are again marked with tiny squares.

However, a strong point in favor of B-splines is that this technique makes it easy to draw a very smooth curve consisting of many curve segments. To avoid confusion, note that each curve segment consists of many straight line segments. For example, Fig. 4.17 shows one curve segment, which consists of 50 line segments. Since four control points are required for a single curve segment we have

$$\text{Number of control points} = \text{Number of curve segments} + 3$$

Figure 4.18 seems to violate this rule, since there are five curve segments, and it looks as if there are only six control points. However, two of these were used twice. This curve was constructed by clicking first on the lower-left vertex (= point 0), followed by clicking on the upper-left one (= point 1), then the upper-right one (= point 2), and so on, following the polygon counter-clockwise. Altogether, the eight control points 0, 1, 2, 3, 4, 5, 0, 1 were selected, in that order. If after this, a key on the keyboard is pressed, only the curve is redrawn, not the control points and the lines that connect them.

If we had clicked on yet another control point (point 2, at the top, right), we would have had a closed curve. In general, to produce a closed curve, there must be three overlapping vertices, that is, two overlapping polygon edges. As you can see in Fig. 4.18, the curve is very smooth indeed: we have second-order continuity, as discussed in the previous section. Recall that this implies that even the curvature is continuous in the points where two adjacent curve segments meet. As the part of the curve near the lower-right corner shows, we can make the distance between a curve and the given points very small by supplying several points close together.

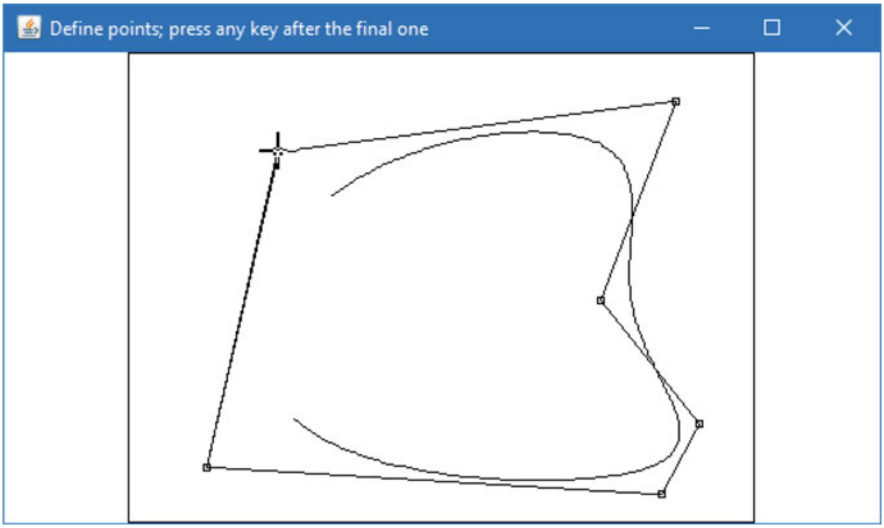


Fig. 4.18 B-spline curve consisting of five curve segments

The mathematics for B-splines can be expressed by the following matrix equation, similar to Eq. (4.20):

$$B(t) = \frac{1}{6}\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{4.21}$$

If we have n control points $P_0, P_1, \ldots, P_{n-1}$, $(n \geq 4)$ then, strictly speaking, Eq. (4.21) applies only to the first curve segment. For the second, we have to replace the points P_0, P_1, P_2 and P_3 in the column vector with P_1, P_2, P_3 and P_4, and so on. As with Bézier curves, the variable t ranges from 0 to 1 for each curve segment. Multiplying the above 4×4 matrix by the column vector that follows it, we obtain

$$B(t) = \frac{1}{6}\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} -P_0 + 3P_1 - 3P_2 + P_3 \\ 3P_0 - 6P_1 + 3P_2 \\ -3P_0 + 3P_2 \\ P_0 + 4P_1 + P_2 \end{bmatrix}$$

or

$$B(t) = \frac{1}{6}(-P_0 + 3P_1 - 3P_2 + P_3)t^3 + \frac{1}{2}(P_0 - 2P_1 + P_2)t^2 +$$
$$\frac{1}{2}(-P_0 + P_2)t + \frac{1}{6}(P_0 + 4P_1 + P_2)$$

The following program is based on this equation. The user can click any number of points, which are used as the points $P_0, P_1, \ldots, P_{n-1}$. The first curve segment appears immediately after the fourth control point, P_3, has been defined, and each additional control point causes a new curve segment to appear. To show only the curve, the user can press any key, which also terminates the input process. After this, we can generate another curve by clicking again. The old curve then disappears. Figures 4.17 and 4.18 have been produced by this program:

```
// Bspline.java: B-spline curve fitting.
// Uses: Point2D (Section 1.4).
import java.awt.*;
import java.awt.event.*;
import java.util.*;

public class Bspline extends Frame {
   public static void main(String[] args) {new Bspline();}

   Bspline() {
      super("Define points; press any key after the final one");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e){
            System.exit(0);
         }
      });
      setSize(500, 300); add("Center", new CvBspline());
      setCursor(Cursor.getPredefinedCursor(
         Cursor.CROSSHAIR_CURSOR));
      setVisible(true);
   }
}

class CvBspline extends Canvas {
   Vector<Point2D> V = new Vector<Point2D>();
   int np = 0, centerX, centerY;
   float rWidth = 10.0F, rHeight = 7.5F, eps = rWidth / 100F,
         pixelSize;
   boolean ready = false;

   CvBspline() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            float x = fx(evt.getX()), y = fy(evt.getY());
            if (ready) {
               V.removeAllElements();
               np = 0;
               ready = false;
            }
```

```
          V.addElement(new Point2D(x, y));
          np++; repaint();
       }
    });

    addKeyListener(new KeyAdapter() {
       public void keyTyped(KeyEvent evt) {
          evt.getKeyChar();
          if (np >= 4) ready = true;
          repaint();
       }
    });
 }

 void initgr() {
    Dimension d = getSize();
    int maxX = d.width - 1, maxY = d.height - 1;
    pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
    centerX = maxX / 2; centerY = maxY / 2;
 }

 int iX(float x){return Math.round(centerX + x / pixelSize);}
 int iY(float y){return Math.round(centerY - y / pixelSize);}
 float fx(int x){return (x - centerX) * pixelSize;}
 float fy(int y){return (centerY - y) * pixelSize;}

 void bspline(Graphics g, Point2D[] p) {
    int m = 50, n = p.length;
    float xA, yA, xB, yB, xC, yC, xD, yD, a0, a1, a2, a3,
       b0, b1, b2, b3, x = 0, y = 0, x0, y0;
    boolean first = true;
    for (int i = 1; i < n - 2; i++) {
       xA = p[i-1].x; xB = p[i].x;
       xC = p[i+1].x; xD = p[i+2].x;
       yA = p[i-1].y; yB = p[i].y;
       yC = p[i+1].y; yD = p[i+2].y;
       a3 = (-xA + 3 * (xB - xC) + xD) / 6;
       b3 = (-yA + 3 * (yB - yC) + yD) / 6;
       a2 = (xA - 2 * xB + xC)/2; b2 = (yA - 2 * yB + yC)/2;
       a1 = (xC - xA) / 2; b1 = (yC - yA) / 2;
       a0 = (xA + 4 * xB + xC)/6; b0 = (yA + 4 * yB + yC)/6;
       for (int j = 0; j <= m; j++) {
          x0 = x; y0 = y;
          float t = (float) j / (float) m;
```

```
            x = ((a3 * t + a2) * t + a1) * t + a0;
            y = ((b3 * t + b2) * t + b1) * t + b0;
            if (first) first = false;
            else
               g.drawLine(iX(x0), iY(y0), iX(x), iY(y));
         }
      }
   }

   public void paint(Graphics g) {
      initgr();
      int left = iX(-rWidth / 2), right = iX(rWidth / 2),
          bottom = iY(-rHeight / 2), top = iY(rHeight / 2);
      g.drawRect(left, top, right - left, bottom - top);
      Point2D[] p = new Point2D[np]; V.copyInto(p);
      if (!ready) {
         for (int i = 0; i < np; i++) {
            // Show tiny rectangle around point:
            g.drawRect(iX(p[i].x) - 2, iY(p[i].y) - 2, 4, 4);
            if (i > 0) // Draw line p[i-1]p[i]:
               g.drawLine(iX(p[i - 1].x), iY(p[i - 1].y),
                          iX(p[i].x), iY(p[i].y));
         }
      }
      if (np >= 4) bspline(g, p);
   }
}
```

To see why B-splines are so smooth, you should differentiate $B(t)$ twice and verify that, for any curve segment other than the final one, the values of $B(1)$, $B'(1)$ and $B''(1)$ at the endpoints of these segments are equal to the values $B(0)$, $B'(0)$ and $B''(0)$ at the start point of the next curve segment. For example, for the continuity of the curve itself we find

$$\begin{aligned} B(1) &= \frac{1}{6}(-P_0 + 3P_1 - 3P_2 + P_3) + \frac{1}{2}(P_0 - 2P_1 + P_2) + \frac{1}{2}(-P_0 + P_2) \\ &\quad + \frac{1}{6}(P_0 + 4P_1 + P_2) \\ &= \frac{1}{6}(P_1 + 4P_2 + P_3) \end{aligned}$$

for the first curve segment, based on P_0, P_1, P_2 and P_3, while we can immediately see that we obtain exactly this value if we compute $B(0)$ for the second curve segment, based on P_1, P_2, P_3 and P_4.

Exercises

4.1 Replace the *drawLine* method based on Bresenham's algorithm and listed almost at the end of Sect. 1 with an even faster version that benefits from the symmetry of the two halves of the line. For example, with endpoints P and Q satisfying Eq. (4.1), and using the integer value x_{Mid} halfway between x_{P} and x_{Q}, we can let the variable x run from x_{P} to x_{Mid} and also use a variable x_2, which at the same time runs backward from x_{Q} to x_{Mid}. In each iteration of the loop, x is increased by 1 and x_2 is decreased by 1. Note that there will be either one point or two points in the middle of the line, depending on the number of pixels to be plotted being odd or even. Be sure that no pixel of the line is omitted and that no pixel is put twice on the screen. To test the latter, you can use XOR mode so that writing the same pixel twice would have the same effect as omitting a pixel.

4.2 Generalize the method *doubleStep*2 (of Sect. 2), to make it work for any lines. The program should allow the user to enter the two endpoints of each line through mouse-clicking. It should take as many lines as the user enters. Make sure to take care of termination condition, i.e. a line with an odd number of pixels should stop just before the last pixel (i.e. the 2nd endpoint) and simply draw the last pixel after exiting the loop.

4.3 Add Bresenham's algorithm into the code of Ex. 4.2 to compare the performance of the two algorithms. For each line specified in Ex. 4.2, your program should draw the line twice, in blue color using Bresenham's algorithm and in red color using the double-step algorithm. Your program should also display the time taken (using the Java method *nanoTime*) for each algorithm. Your timer should start as the first statement (after declarations) and finish as the last statement inside the implemented *drawLine* method. The displayed times may read like

"Bresenham: xxx ns"
"Double-Step: xxx ns"

on a separate area of the canvas, not on the command line.

4.4 Since normal pixels are very small, they do not show very clearly which of them are selected by Bresenham's algorithms. Use a grid to simulate a screen with a very low resolution. Demonstrate both the method *drawLine* (with *g* as its first argument) of Sect. 1 and the method *drawCircle* of Sect. 3. Only the gridpoints of your grid are to be used as the centers of 'superpixels'. The method *putPixel* is to draw a small circle with such a center and the distance *dGrid* of two neighboring gridpoints as its diameter. Do not change the methods *drawLine* and *drawCircle* that we have developed, but use *dGrid*, just mentioned, in a method *putPixel* that is very different from the one shown at the beginning of Sect. 1. Figure 4.19 shows a grid (with *dGrid* = 10) and both a line and a circle drawn in this way. As in Fig. 4.1, the line shown here has the endpoints P(1, 1) and Q(12, 5) but this time the positive y-axis points

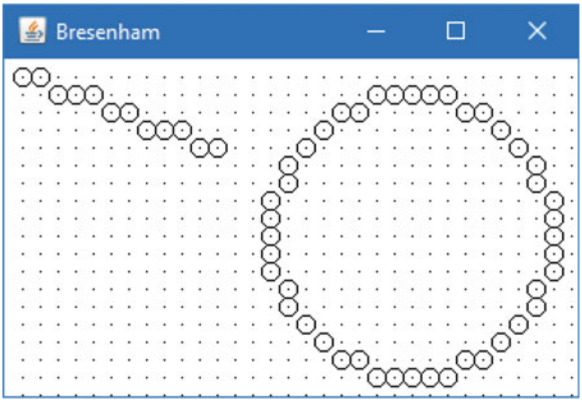


Fig. 4.19 Bresenham algorithms for a line and for a circle (see also Figs. 4.1 and 4.6)

downward and the origin is the upper-left corner of the drawing rectangle. The circle has radius $r = 8$, and is approximated by the same pixels as shown in Fig. 4.6 for one eighth of this circle. The line and circle were produced by the following calls to the methods *drawLine* and *drawCircle* of Sects. 1 and 3 (but with a different method *putPixel*):

```
drawLine(g, 1, 1, 12, 5);   // g, xP, yP, xQ, yQ
drawCircle(g, 23, 10, 8);   // g, xC, yC, r
```

4.5 In the *Bezier.java* we constructed only one curve segment, using the points P_0, P_1, P_2 and P_3. Extend this program to enable the user to draw very smooth curves consisting of more than one segment. After the first segment has been drawn, the second one is based on these points Q_0, Q_1, Q_2 and Q_3 (similar to P_0, P_1, P_2 and P_3):

- Q_0 is identical with P_3; clicking on P_3 acts as a signal that we want to start another curve segment.
- Q_1 is not specified by the user, but is automatically constructed by reflecting P_2 about P_3; in other words, P_3 (= Q_0) will be the midpoint of P_2Q_1. This principle will make the curve very smooth.
- Q_2 and Q_3 are defined in the usual way, that is, by clicking on these points.

It should also be possible to specify a third curve segment by starting at Q_3, and so on. When the curve (consisting of an arbitrary number of segments) is completed, the user clicks on a point other than the final one of the last segment. This will be the starting point (similar of P_0) of an entirely new curve (to be drawn in addition to the previous one), unless, after this, the user clicks on this new point once again: this clicking twice on the same point will be the signal that the drawing is ready. As soon as a curve is completed, it is to be drawn without displaying any straight lines connecting control points and without any little squares marking these points.

4.6 Extend the program *Bspline.java* of Sect. 7. Supply a grid, with visible gridpoints lying, say, 10 pixels apart, horizontally and vertically. Any control point specified by the user using the mouse should be pulled to the nearest gridpoint. This makes it easier for the user to specify two or more points that lie on the same horizontal or vertical line. Apart from using a grid, your program will also be different from *Bspline.java* in that it must be able to store several B-spine curves instead of only one. The following characters are to be interpreted as commands:

+ Increase the distance between gridpoints by one.
– Decrease the distance between gridpoints by one.
n After this command, start with a new curve, retaining the previous ones.
d Delete the last curve.
g Change the visibility status of gridpoints (visible/invisible).
c Show only the curves, without any control points or lines connecting these.

4.7 Implement a restricted version of Bresenham's line-drawing algorithm as a demonstration. The purpose of the exercise is to visualize how this algorithm works by showing its stepwise execution on an exaggerated screen area as a 10×10 grid. Draw the grid on the left half of the drawing space and the main body of method *drawLine*2 (listed in Sect. 1) on the right half with text font size of 14. At the bottom of the drawing space, add both a button labeled *Step* and a small space for displaying the values of the program variables *dx*, *c*, *m*, *d*, *x* and *y*. When the user clicks on two intersection points P and Q, satisfying Eq. (4.1), on the grid, a thin line is drawn and the initial values of the program variables *d* and *m* are displayed. Just after the definition of point Q, the call to *putPixel* is highlighted (by coloring or boxing the statement), indicating that this program line is next to be executed. Then whenever the user clicks on *Step*, the highlighted program line is executed, the next program line in the right half is highlighted, and the values of *d*, *x* and *y* are updated. Each time *putPixel* is executed, a filled circle is drawn on the appropriate grid intersection (representing a pixel on the line).

Chapter 5
Perspective and 3D Data Structure

We now turn to the exciting subject of 3D graphics. As soon as we know how to compute the perspective image of a single point, we can easily produce more interesting images. To obtain the perspective image of a straight line, we simply connect the images of its endpoints, using the fact that the image of a straight line is also a straight line. In this chapter, the computation of the perspective image of a point is done in two steps: a viewing transformation followed by a perspective transformehation.

5.1 Introduction

In Fig. 5.1 a two-dimensional representation of a cube is shown along with some auxiliary lines. Although AB is a horizontal edge, it is not a horizontal line in the picture. Lines in 3D space that are horizontal and parallel meet in the picture in a so-called *vanishing point*. All these vanishing points lie on the same line, which is called the *horizon*. Horizon and vanishing points refer to the 2D image space, not to the 3D object space. For many centuries these concepts have been used by artists to draw realistic images of three-dimensional objects. This way of representing three-dimensional objects is usually referred to as *perspective*.

The invention of photography offered a new (and easier) way of producing images in perspective. There is a strong analogy between a camera used in photography and the human eye. Our eye is a very sophisticated instrument of which a camera is an imitation. In the following discussion the word *eye* may be replaced with *camera*.

It is obvious that the image will depend upon the position of the eye. An important aspect is the distance between the eye and the object, since the effect of perspective will be inversely proportional to this distance. If the eye is close to the object, the effect of perspective is strong, as shown in Fig. 5.2a. Here we can very clearly see that in the image the extensions of parallel line segments meet.

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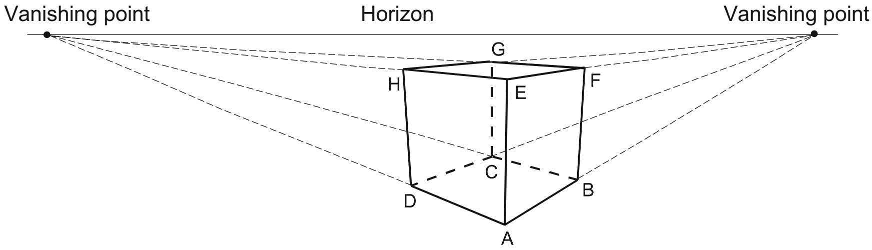


Fig. 5.1 Vanishing points on the horizon

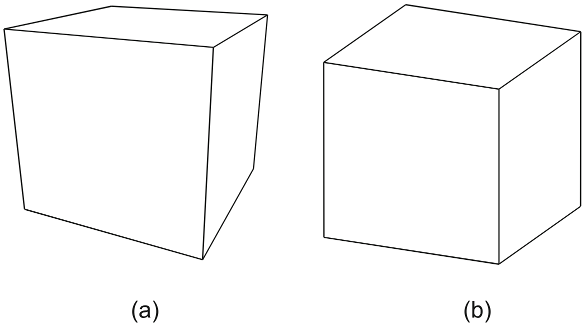


Fig. 5.2 (**a**) Eye nearby and (**b**) eye far away

Besides the classical and the photographic method, there is a way of producing perspective images based on analytical geometry. Let us write X and Y for 2D and x, y and z for 3D coordinates.

If we want to produce a drawing in perspective, we are given a great many points P(x, y, z) of the object and we want their images P′(X, Y) in the picture. Thus all we need is a mapping from the *world coordinates* (x, y, z) of a point P to the *screen coordinates* (X, Y) of its central projection P′. We imagine a screen between the object and the eye E. For every point P of the object the line PE intersects the screen at point P′. It is convenient to perform this mapping in two stages. The first is called a *viewing transformation*; point P is left at its place, but we change from world coordinates to so-called *eye coordinates*. The second stage is called a *perspective transformation*. This is a proper transformation from P to P′, combined with a transition from the three-dimensional eye coordinates to the two-dimensional screen coordinates:

$$\begin{array}{rl} & \text{World coordinates } (x_{\mathrm{w}}, y_{\mathrm{w}}, z_{\mathrm{w}}) \\ \text{Viewing transformation} \downarrow & \\ & \text{Eye coordinates } (x_{\mathrm{e}}, y_{\mathrm{e}}, z_{\mathrm{e}}) \\ \text{Perspective transformation} \downarrow & \\ & \text{Screen coordinates } (X, Y) \end{array}$$

5.2 Viewing Transformation

To perform the viewing transformation we must be given an object and a viewpoint E. Let us require that the world-coordinate system be right-handed. It is convenient if its origin O lies more or less centrally in the object; we then view the object from E to O. We will assume this to be the case; in practice this might require a coordinate transformation consisting of decreasing the original world coordinates by the coordinates of the central object point. We will include this very simple coordinate transformation in our program, without writing it down in mathematical notation.

Let the viewpoint E be given by its spherical coordinates ρ (= *rho*), θ (= *theta*), φ (= *phi*), relative to the world-coordinate system. Thus its world-coordinates are

$$\begin{aligned} x_\mathrm{E} &= \rho \sin \varphi \cos \theta \\ y_\mathrm{E} &= \rho \sin \varphi \sin \theta \\ z_\mathrm{E} &= \rho \cos \varphi \end{aligned} \tag{5.1}$$

as shown in Fig. 5.3.

The direction of vector **EO** (= −**OE**) is said to be the *viewing direction*. From our eye at E we can only see points within some cone whose axis is EO and whose apex is E. If the Cartesian coordinates x_E, y_E, z_E of viewpoint E were given, we could derive the spherical coordinates from them as follows:

```
rho = Math.sqrt(xE * xE + yE * yE + zE * zE);
theta = Math.atan2(yE, xE);
phi = Math.acos(zE/rho);
```

Our final objective will be to compute the screen coordinates X, Y, where we have an X-axis and a Y-axis, lying in a screen between E and O and perpendicular to

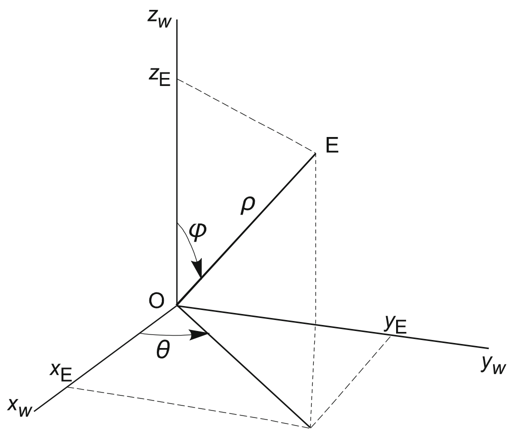


Fig. 5.3 Spherical coordinates of viewpoint E

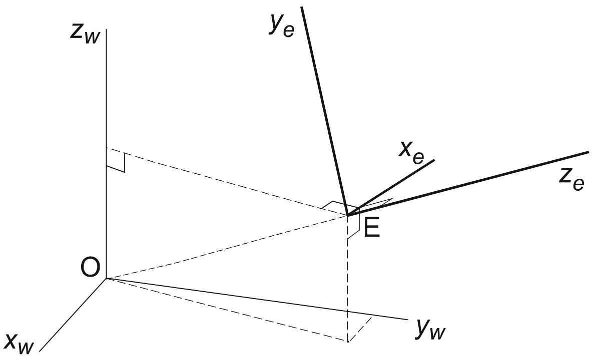


Fig. 5.4 Eye-coordinate system

the viewing direction **EO**. This is why the eye-coordinate system, which we will deal with first, will have its x_e-axis and y_e-axis perpendicular to **EO**, leaving the z_e-axis in the direction of OE. The origin of the eye-coordinate system is viewpoint E, as shown in Fig. 5.4. Viewing from E to O, we find the positive x_e-axis pointing to the right and the positive y_e-axis upwards. These directions will later enable us to establish screen axes in the same directions. We could have used a positive z_e-axis pointing from E to O; on the one hand this is attractive because it makes the z_e-coordinates of all object points positive, but, on the other, it would have required a left-handed eye-coordinate system. In this book we will use a right-handed eye-coordinate system (as shown in Fig. 5.4) to avoid confusion with regard to the use of the cross product, taking the minus sign of z_e-coordinates into the bargain.

The viewing transformation can be written as a matrix multiplication, for which we need the 4 × 4 *viewing matrix* V:

$$[x_e \quad y_e \quad z_e \quad 1] = [x_w \quad y_w \quad z_w \quad 1]V \tag{5.2}$$

To find V, we imagine this transformation to be composed of three elementary ones, for which the matrices can easily be written down. Matrix V will be the product of these three matrices. Each of the three transformations is in fact a change of coordinates and has therefore a matrix which is the inverse of the matrix for a similar point transformation.

1. **Moving the origin from O to E**

We perform a translation of the coordinate system such that viewpoint E becomes the new origin. The matrix for this change of coordinates is

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_E & -y_E & -z_E & 1 \end{bmatrix} \tag{5.3}$$

(Do not confuse x_E, y_E, z_E, the world coordinates of viewpoint E, with x_e, y_e, z_e, the eye coordinates of any point.) The new coordinate system is shown in Fig. 5.5.

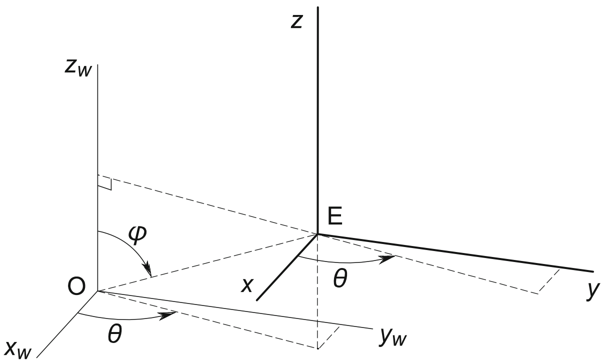


Fig. 5.5 Situation before rotation about the z-axis

2. **Rotating the coordinate system about the z-axis**

Starting with Fig. 5.5 we rotate the coordinate system about the z-axis through the angle $\theta + 90°$, so the new x-axis points to the right and is perpendicular to the vertical plane through E and O. The matrix for this change of coordinates is the same as that for a rotation of points through the angle $-(\theta + 90°)$, that is, $-\theta - 90°$. We obtain the 4 × 4 matrix R_z for this rotation by using a 2 × 2 matrix for this purpose, as discussed in Sect. 3.2, and adding the third and fourth columns and rows of a 4 × 4 unit matrix:

$$R_z = \begin{bmatrix} \cos\left(-\theta - 90^\circ\right) & \sin\left(-\theta - 90^\circ\right) & 0 & 0 \\ -\sin\left(-\theta - 90^\circ\right) & \cos\left(-\theta - 90^\circ\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.4}$$

If you find the simplifications

$$\cos\left(-\theta - 90^\circ\right) = -\sin\theta \quad \text{and} \quad \sin\left(-\theta - 90^\circ\right) = -\cos\theta$$

difficult, it will be helpful to plot the two angles of this formula in a unit circle for some value of θ, as Fig. 5.6 illustrates.

After applying the above matrix R_z, the new position of the x-, y- and z-axes is as shown in Fig. 5.7.

3. **Rotating the coordinate system about the x-axis**

Since the z-axis is to have the direction **OE**, we now rotate the coordinate system about the x-axis through the angle φ. The dashed line near the positive z-axis in

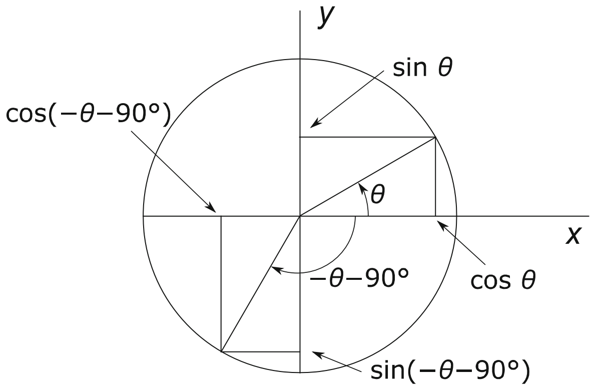


Fig. 5.6 Relating the sine and cosine of −θ − 90° to those of θ

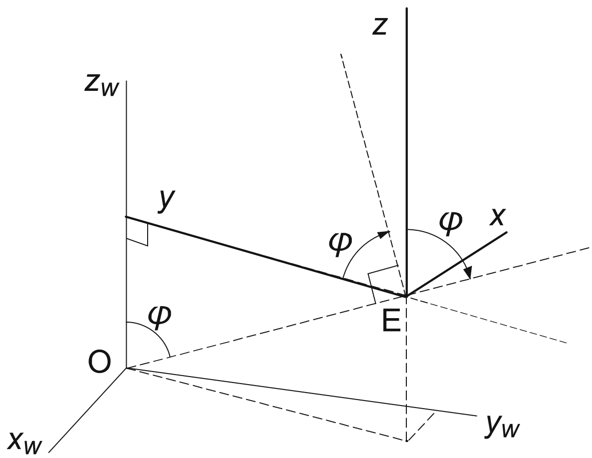


Fig. 5.7 Situation before rotation about x-axis

Fig. 5.7 indicates the new y-axis after this rotation. A rotation about the x-axis, in such a way that the y-axis goes towards the z-axis is a positive one: it corresponds to a right-handed screw moving into the direction of the positive x-axis. However, since we are performing a coordinate transformation instead of rotating points, we have to use $-\varphi$ instead of φ as the angle of rotation, so that we obtain the following rotation matrix:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\varphi) & \sin(-\varphi) & 0 \\ 0 & -\sin(-\varphi) & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.5}$$

After this final rotation, we have obtained the eye-coordinate system with x_e-, y_e- and z_e-axes, which we have already seen in Fig. 5.4. Multiplying the above matrices T, R_z and R_x, we obtain the desired viewing matrix:

$$V = TR_zR_x = \begin{bmatrix} -\sin\theta & -\cos\varphi\cos\theta & \sin\varphi\cos\theta & 0 \\ \cos\theta & -\cos\varphi\sin\theta & \sin\varphi\sin\theta & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & -\rho & 1 \end{bmatrix} \tag{5.6}$$

Recall that we use this matrix in Eq. (5.2), to compute the eye coordinates x_e, y_e and z_e from the given world coordinates x_w, y_w and z_w.

The viewing transformation, which we have now dealt with, is to be followed by the perspective transformation to be discussed in the next section. However, we could also use the eye coordinates x_e and y_e, simply ignoring z_e. In that case we have a so-called *orthographic projection*. Every point P of the object is then projected into a point P′ by drawing a line from P, perpendicular to the plane through the x-axis and the y-axis. It can also be regarded as the perspective image we obtain if the viewpoint is infinitely far away. An example of such a picture is the cube in Fig. 5.2b. Parallel lines remain parallel in pictures obtained by orthographic projection. Such pictures are very often used in practice because with conventional methods they are easier to draw than real perspective images.

On the other hand, bringing some perspective into the picture will make it much more realistic. Our viewing transformation will therefore be followed by the perspective transformation, which will involve surprisingly little computation.

5.3 Perspective Transformation

You might have the impression that we are only half-way, and that in this section we will need as much mathematics as in Sect. 5.2. However, most of the work has already been done. Since we will not use world coordinates in this section, there will be no confusion if we denote eye coordinates simply by (x, y, z) instead of (x_e, y_e, z_e).

In Fig. 5.8 we have chosen a point Q, whose eye coordinates are $(0, 0, -d)$ for some positive value d.

Our screen will be the plane $z = -d$, that is, the plane through Q and perpendicular to the z-axis. Then the screen-coordinate system has Q as its origin, and its X- and Y-axes are parallel to the x- and y-axes. For every object point P, the image point P′ is the intersection of line PE and the screen. To keep Fig. 5.8 simple, we consider a point P whose y-coordinate is zero. However, the following equations to compute its screen coordinate X are also valid for other y-coordinates. In Fig. 5.8 the triangles EPR and EP′Q are similar. Hence

$$\frac{\mathrm{P'Q}}{\mathrm{EQ}} = \frac{\mathrm{PR}}{\mathrm{ER}}$$

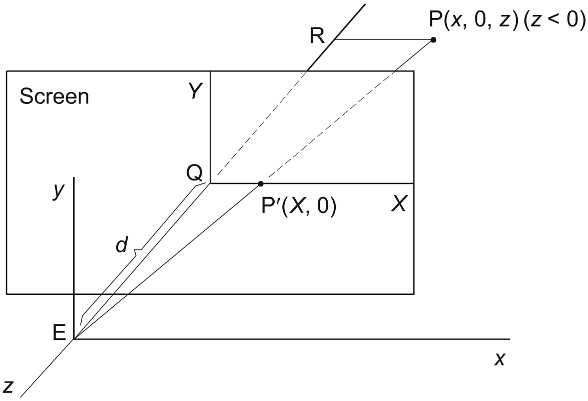


Fig. 5.8 Screen and eye coordinates

so we have

$$\frac{X}{d} = \frac{x}{-z}$$

(Recall that z-coordinates of object points are negative, so that $-z$ is a positive value.) In other words,

$$X = -d \cdot \frac{x}{z} \tag{5.7}$$

In the same way we can derive

$$Y = -d \cdot \frac{y}{z} \tag{5.8}$$

At the beginning of Sect. 5.2 we have chosen the origin O of the world-coordinate system to be a central point of the object. The origin Q of the screen-coordinate system will be central in the image because the z-axis of the eye-coordinate system is a line through E and O, which intersects the screen at Q. We must bear in mind that Eqs. (5.7 and 5.8) can be used in this form only if the origin Q of the screen coordinate system (with X- and Y-axes) lies in the center of the screen. If this origin lies instead in the lower-left corner of the screen and the screen has width w and height h, we have to add $w/2$ and $h/2$ to Eqs. (5.7 and 5.8), respectively.

We still have to specify the distance d between viewpoint E and the screen. Roughly speaking, we have

$$\frac{d}{\rho} = \frac{\text{image size}}{\text{object size}}$$

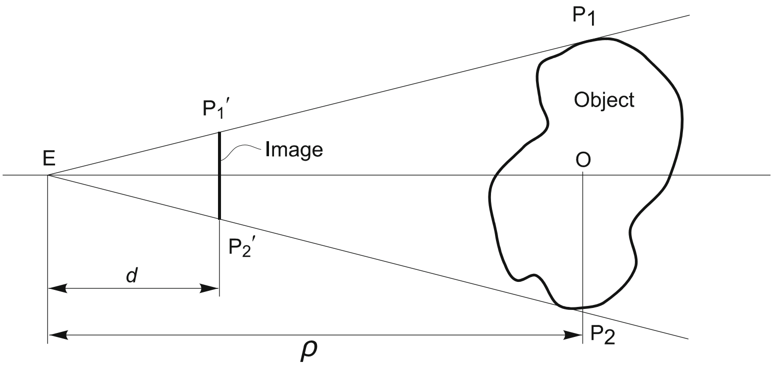


Fig. 5.9 Image size and object size

which follows from the similarity of the triangles $EP_1'P_2'$ and EP_1P_2 in Fig. 5.9. Thus we have

$$d = \rho \cdot \frac{\text{image size}}{\text{object size}} \tag{5.9}$$

This equation should be applied to both the horizontal and the vertical directions. It should be interpreted only as a means to obtain an indication about an appropriate value for d. A three-dimensional object may have a complicated shape, and it may not be clear how its size is to be measured. We then use a rough estimation of the object size, such as the maximum of its length, width and height. The image size in Eq. (5.9) should be taken somewhat smaller than the screen.

5.4 A Cube in Perspective

We will now discuss a complete Java program, which draws a perspective representation of a cube, as shown in Fig. 5.10. Such representations, with all edges visible, are called *wire-frame* models.

To specify this cube in a program, we assign numbers to its vertices, as shown in Fig. 5.11. The center of the cube coincides with the origin O and its edges have length 2, which implies that the x-, y- and z-coordinates of its eight vertices are equal to +1 or -1

The following program produces the wire-frame model of Fig. 5.10. We store the world, eye and screen coordinates for each of the eight vertices of the cube in the class *Obj*:

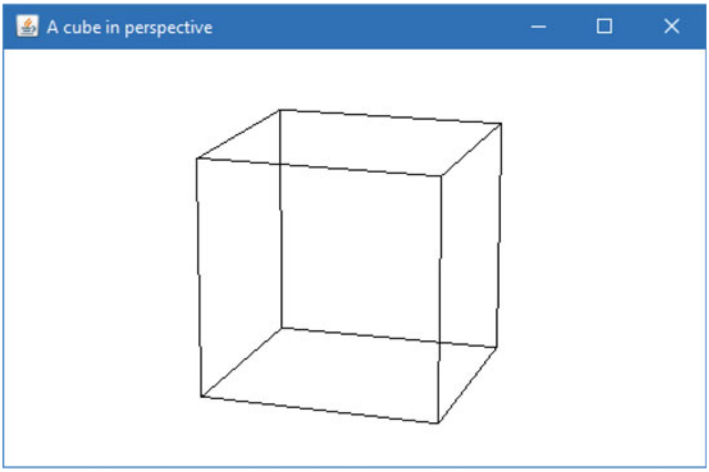
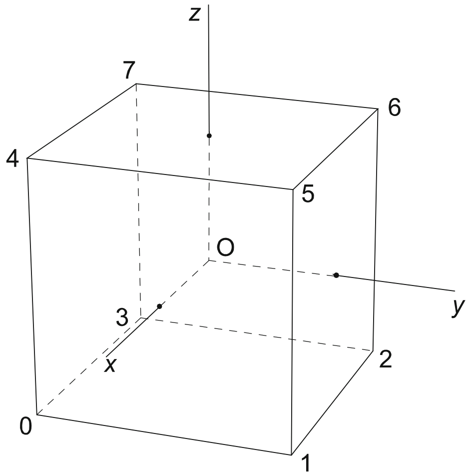


Fig. 5.10 Output of program *CubePers.java*

Fig. 5.11 Vertex numbers and coordinate axes




```
// CubePers.java: A cube in perspective.
// Uses: Point2D (Section 1.4), Point3D (Section 3.9).
import java.awt.*;
import java.awt.event.*;

public class CubePers extends Frame {
   public static void main(String[] args) {new CubePers();}

   CubePers() {
      super("A cube in perspective");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
```

```
      setLayout(new BorderLayout());
      add("Center", new CvCubePers());
      Dimension dim = getToolkit().getScreenSize();
      setSize(dim.width / 2, dim.height / 2);
      setLocation(dim.width / 4, dim.height / 4);
      setVisible(true);
   }
}

class CvCubePers extends Canvas {
   int centerX, centerY;
   Obj obj = new Obj();

   int iX(float x) {
      return Math.round(centerX + x);
   }

   int iY(float y) {
      return Math.round(centerY - y);
   }

   void line(Graphics g, int i, int j) {
      Point2D p = obj.vScr[i], q = obj.vScr[j];
      g.drawLine(iX(p.x), iY(p.y), iX(q.x), iY(q.y));
   }

   public void paint(Graphics g) {
      Dimension dim = getSize();
      int maxX = dim.width - 1, maxY = dim.height - 1,
          minMaxXY = Math.min(maxX, maxY);
      centerX = maxX / 2; centerY = maxY / 2;
      obj.d = obj.rho * minMaxXY / obj.objSize;
      obj.eyeAndScreen();
      // Horizontal edges at the bottom:
      line(g, 0, 1); line(g, 1, 2); line(g, 2, 3); line(g, 3, 0);
      // Horizontal edges at the top:
      line(g, 4, 5); line(g, 5, 6); line(g, 6, 7); line(g, 7, 4);
      // Vertical edges:
      line(g, 0, 4); line(g, 1, 5); line(g, 2, 6); line(g, 3, 7);
   }
}
```

```
class Obj { // Contains 3D object data
   float rho, theta = 0.3F, phi = 1.3F, d, objSize,
         v11, v12, v13, v21, v22, v23, v32, v33, v43;
                   // Elements of viewing matrix V
   Point3D[] w;    // World coordinates
   Point2D[] vScr; // Screen coordinates

   Obj() {
      w = new Point3D[8];
      vScr = new Point2D[8];
      // Bottom surface:
      w[0] = new Point3D(1, -1, -1);
      w[1] = new Point3D(1, 1, -1);
      w[2] = new Point3D(-1, 1, -1);
      w[3] = new Point3D(-1, -1, -1);
      // Top surface:
      w[4] = new Point3D(1, -1, 1);
      w[5] = new Point3D(1, 1, 1);
      w[6] = new Point3D(-1, 1, 1);
      w[7] = new Point3D(-1, -1, 1);
      objSize = (float) Math.sqrt(12F);
      // = sqrt(2 * 2 + 2 * 2 + 2 * 2)
      // = distance between two opposite vertices.
      rho = 5 * objSize; // For reasonable perspective effect
   }

   void initPersp() {
      float costh = (float) Math.cos(theta),
            sinth = (float) Math.sin(theta),
            cosph = (float) Math.cos(phi),
            sinph = (float) Math.sin(phi);
      v11 = -sinth; v12 = -cosph * costh;  v13 = sinph * costh;
      v21 = costh;  v22 = -cosph * sinth;  v23 = sinph * sinth;
                    v32 = sinph;           v33 = cosph;
                                           v43 = -rho;
   }

   void eyeAndScreen() {
      initPersp();
      for (int i = 0; i < 8; i++) {
         Point3D p = w[i];
         float  x = v11 * p.x + v21 * p.y,
                y = v12 * p.x + v22 * p.y + v32 * p.z,
                z = v13 * p.x + v23 * p.y + v33 * p.z + v43;
```

```
            vScr[i] = new Point2D(-d * x / z, -d * y / z);
        }
    }
}
```

As discussed in Sect. 5.3 (see point Q in Fig. 5.8), the perspective transformation is simplest if we use a coordinate system with the origin in the center of the screen and with a y-axis pointing upward. To convert such (floating-point) coordinates to the low-level Java coordinates, of type *int*, we again use the methods *iX* and *iY*. As usual, these are based on the values of *centerX*, *centerY* and *maxY*, which are computed in the *paint* method before this method uses *iX* and *iY*.

The *paint* method also computes the screen distance d (stored in *obj*), using Eq. (5.9). After calling the *Obj* method *eyeAndScreen* to compute eye and screen coordinates, we use our method *line* to draw all 12 cube edges.

In the class *Obj*, the arrays w and *vScr* contain the world and the screen coordinates, respectively, of the cube vertices. Recall that we use the matrix multiplication of Eq. (5.2) to compute the eye coordinates from the given world coordinates stored in w. We then use these eye coordinates to compute the screen coordinates for the array *vScr*. Figure 5.11 is helpful in specifying the coordinate values stored in the array w and the vertex numbers used in calls to the *line* method.

5.5 Specification and Representation of 3D Objects

We will now discuss the way we will specify and represent 3D objects. In most cases these will be solid objects such as a cube, or a sphere approximated by a polyhedron, and we will use their polygonal faces to specify them. We will also show how to specify (a) holes in such faces, (b) line segments that are not necessarily edges of faces, and (c) polygons that are not boundary faces of solid objects and therefore, depending on the viewpoint, visible on both sides.

In all these cases, we will use input files consisting of two parts: a list of vertices, each in the form of a nonnegative vertex number followed by three real numbers, the world coordinates of that vertex. The second part consists of an input line of the form

```
Faces:
```

followed by sequences of vertex numbers, each sequence followed by a period '.'.

Let us first deal with the most common case, with each such sequence consisting of three or more vertex numbers specifying a polygon that is a face of a solid 3D object. For each polygon, when viewed from outside the object, the vertex sequence must be counter-clockwise. This orientation must also apply to the first three vertices of each sequence; in other words, the second number of each sequence must denote a convex vertex. The files we are discussing can be used as input for four programs in this book: *Wireframe* (Sect. 5.7), *Hlines* (Sect. 6.1), *Painter* (Sect. 6.3), and *ZBuf* (Sect. 6.4).

Our first example is the cube of Fig. 5.11, with the origin O in its center. With vertex numbers as shown in the figure, we can specify this cube as follows:

```
0   1 -1 -1
1   1  1 -1
2  -1  1 -1
3  -1 -1 -1
4   1 -1  1
5   1  1  1
6  -1  1  1
7  -1 -1  1
Faces:
0 1 5 4.
1 2 6 5.
2 3 7 6.
3 0 4 7.
4 5 6 7.
3 2 1 0.
```

In the first part, the vertex numbers need not be in ascending order, and neither need they be consecutive. We might increase the vertex numbers 4, 5, 6, 7 by 10, put the lines for these vertices at the top, followed by the those for the other vertices with numbers multiplied by 10, giving the following file, which specifies the same 3D object and gives the same graphical result when supplied to the four programs mentioned above:

```
40   1 -1  1
50   1  1  1
60  -1  1  1
70  -1 -1  1
10   1 -1 -1
11   1  1 -1
12  -1  1 -1
13  -1 -1 -1
Faces:
10 11 50 40.
11 12 60 50.
12 13 70 60.
13 10 40 70.
40 50 60 70.
13 12 11 10.
```

The above original specification used a vertex number 0. It may be a good idea to avoid this, as we will see shortly. From now on the lowest vertex number used in our examples will be 1, not 0.

Holes and Invisible Line Segments

It is often desirable to accept object faces that contain holes. For example, consider the solid letter A of Fig. 5.12, the front face of which is not a proper polygon because there is a triangular hole in it. The same applies to the (identical) face on the back. Each vertex i of the front face is connected with vertex $i + 10$ of the back face ($1 \le i \le 10$).

We can turn the front face into a polygon by introducing a very narrow gap, say, between the vertices 7 and 10, as shown in Fig. 5.13. After doing this, we could try to specify this new polygon as.

```
1  2  3  4  5  6  7  10  9  8  10  7.
```

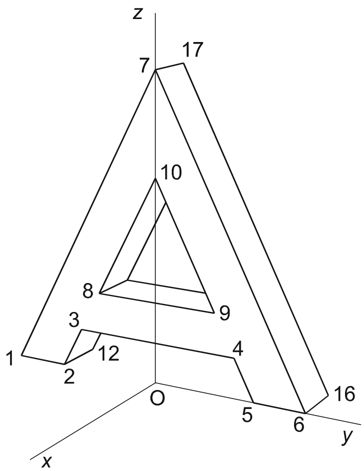


Fig. 5.12 Solid letter A

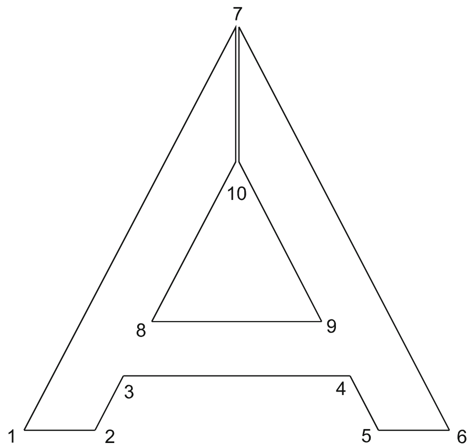


Fig. 5.13 A polygon

Note that this requires the gap (7, 10) to have width zero, so that there is only one vertex (7) at the top. On the other hand, only a real gap makes it clear that the vertex numbers (10, 9, 8) occur in that order in the above input line: just follow the vertices in Fig. 5.13, starting at vertex 1.

If we really specified the front face in the above way, the line (7, 10) would be regarded as a polygon edge and therefore appear as a line in the output of the programs *Wireframe* of Sect. 5.7 and *HLines* of the next chapter. This is clearly undesirable. To prevent this from happening, we adopt the convention of writing a minus sign in front of the second vertex of a pair, indicating that this pair denotes a line segment that, although being an edge of a polygon, is not to be drawn. We do this with the ordered pairs (7, 10) and (10, 7) in the above input line, writing (7, −10) and (10, −7), so that we use the following input line instead of the above one:

```
1  2  3  4  5  6  7  -10  9  8  10  -7.
```

It will now be clear why it makes sense to avoid using 0 as a vertex number: line segment (7, 10) is not drawn here because of the negative value in (7, −10), but, since $-0 = 0$, there would have been no such negative value in this number pair if we had used 0 instead of 10.

The solid letter A of Fig. 5.12 (without vertex numbers and axes) is obtained by applying program *HLines* of the next chapter to the following input file *letterA.dat*, in which the extra minus signs occur in the first two lines after the word *Faces*:

```
1     0 -30   0
2     0 -20   0
3     0 -16   8
4     0  16   8
5     0  20   0
6     0  30   0
7     0   0  60
8     0 -12  16
9     0  12  16
10    0   0  40
11  -10 -30   0
12  -10 -20   0
13  -10 -16   8
14  -10  16   8
15  -10  20   0
16  -10  30   0
17  -10   0  60
18  -10 -12  16
19  -10  12  16
20  -10   0  40
Faces:
```

```
1 2 3 4 5 6 7 -10 9 8 10 -7.
11 17 -20 18 19 20 -17 16 15 14 13 12.
2 12 13 3.
3 13 14 4.
15 5 4 14.
8 9 19 18.
8 18 20 10.
19 9 10 20.
6 16 17 7.
11 1 7 17.
11 12 2 1.
15 16 6 5.
```

(Note that this use of minus signs applies only to vertex numbers in the second part of an input file. In the first part, minus signs can only occur in coordinate values, where they have their usual meaning.)

Implementing this idea is very simple. For example, because of the minus sign that precedes vertex number 10 in

```
... 7  -10 ...
```

we do not store line segment (7, 10) in the data structure that will be discussed later. In other respects, we simply ignore these minus signs. Therefore the set of triangles resulting from the above complete set of input data (for the solid letter A) is the same as when there had been no minus signs in front of any vertex numbers.

Besides for holes, we can also use these minus signs for the approximation of curved surfaces, omitting drawn polygon diagonals in the image, as we will see in Appendix D.

Individual Faces and Line Segments

Although we usually draw polygons that are boundary faces of solid objects, we sometimes want to draw very thin (finite) planes, here also called *faces*. Examples are sheets of paper and a cube made of very thin material, of which the top face is removed, as shown in Fig. 5.14.

Since such faces have two visible sides, we specify each face twice: counter-clockwise for the side we are currently viewing and clockwise for the side that is currently invisible but may become visible when we change the viewpoint. For example, the front face of the cube of Fig. 5.14 is specified twice in the input file:

```
1 2 3 4.
4 3 2 1.
```

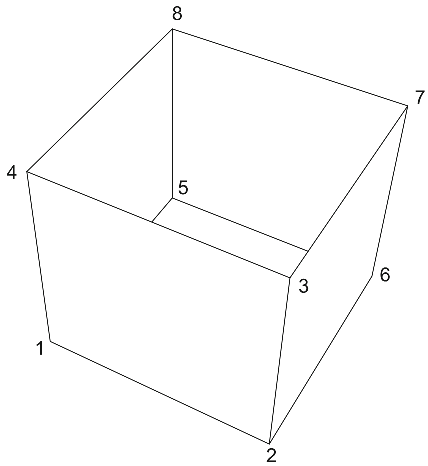


Fig. 5.14 A hollow cube

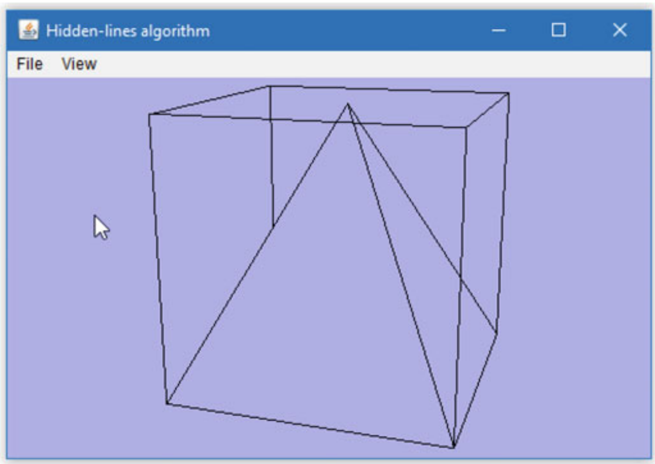


Fig. 5.15 Solid pyramid in wire-frame cube

Although the user supplies polygons in input files as object faces, we deal primarily with line segments, referred to as PQ in the previous section. Besides the polygons and the triangles resulting from them, we also store the edges of the polygons as line segments. It is also desirable to be able to draw line segments that are not edges of polygons.

Examples of such ‘loose’, individual line segments are the axes of a 3D coordinate system. Sometimes we want to define the edges of polygons as individual line segments, to prevent such polygons from obscuring other line segments, displaying objects as wire-frame models. An example of this is shown in Fig. 5.15. Here we have a solid pyramid fitting into a cube. Obviously, the pyramid would not be

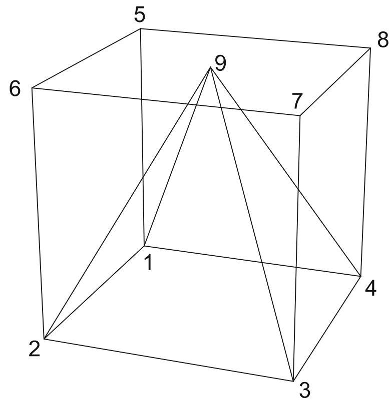


Fig. 5.16 Vertex numbers of pyramid and cube

visible if the cube was solid; we therefore prefer the latter to be displayed as a wireframe model. To provide an input file for this pyramid, we begin by assigning vertex numbers, as shown in Fig. 5.16.

These vertex numbers occur in the following input file:

```
1   0 0 0
2   2 0 0
3   2 2 0
4   0 2 0
5   0 0 2
6   2 0 2
7   2 2 2
8   0 2 2
9   1 1 2
Faces:
1 4 3 2.
1 2 9.
2 3 9.
3 4 9.
4 1 9.
1 5.
2 6.
3 7.
4 8.
5 6.
6 7.
7 8.
8 5.
```

After the word *Faces*, we begin with the square bottom face and the four triangular faces of the pyramid. After this, four vertical and four horizontal cube

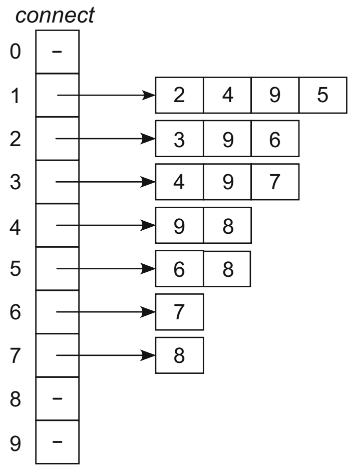


Fig. 5.17 Internal representation of line segments

edges follow, each specified by its two endpoints. It follows that the word *Faces* in our input files should not be taken too literally: this *Faces* section may include pairs of vertex numbers, which are not faces at all but line segments not necessarily belonging to faces.

Since line segments can occur not only as edges of faces but also individually, we will use the array *connect* of objects of the Java class *Vector* to store them in our program. Each line segment will be stored only once. For example, the edge 3-9 of the pyramid of Fig. 5.16 is part of the faces 2-3-9 and 3-4-9, but it would be inefficient to draw it twice. By using a special data structure for line segments, we can ensure that this edge is stored only once. This array *connect* for our example of a pyramid in a cube is shown in Fig. 5.17.

An array element *connect*[*i*] referring to an *Vector* object containing the integer *j* implies that there is a line segment (i, j) to be drawn. By requiring that *i* is less than *j*, we ensure that each line segment is stored only once. For example, *connect* [1] refers to the array containing the integers 2, 4, 9 and 5. This indicates that the following line segments start at vertex 1 (each ending at a vertex that has a number higher than 1): 1-2, 1-4, 1-9 and 1-5, which is in accordance with Fig. 5.16. The next element, *connect*[2] refers to three vertex numbers, 3, 9 and 6. Although, besides 2-3, 2-9 and 2-6, there is also a line segment 2–1 (see Fig. 5.16), this is not included here because 2 is greater than 1 and this segments has already been stored as line segment 1-2.

5.6 Some Useful Classes

To avoid duplication of code, we will now discuss some classes (*Input*, *Obj*3*D*, *Tria*, *Polygon*3*D*, *Canvas*3*D* and *Fr*3*D*) that we will frequently use later.

Input*: A Class for File Input Operations*

The first class we will discuss is *Input*. It is not really specific for computer graphics, but useful for any programming task that involves reading data from a text file in a simple way. We will use it to read data for 3D objects, specified in a particular format. There is an *Input* constructor that accepts the name of an input file as an argument, as well as a constructor without any arguments to read data from the keyboard. To demonstrate how easy it is to use this class, let us suppose we are given a text file containing only numbers and that we want to write a program that reads this file to compute the sum of these numbers. For example, this text file, say, *example.txt*, may have the following contents:

```
2.5     6
200     100
```

Then the desired program is shown below:

```
// Sum.java: Demonstrating the class Input by computing the sum
//    of all numbers in the text file example.txt (which contains
//    only numbers and whitespace characters).

public class Sum {
   public static void main(String[] args) {
      float x, s=0;
      Input inp = new Input("example.txt");
      for (;;) {
         x = inp.readFloat();
         if (inp.fails()) break;
         s += x;
      }
      System.out.println("The computed sum is " + s);
      if (!inp.eof())
         System.out.println("Input file missing or incorrect.");
   }
}
```

After compiling and executing this program in a directory that also contains the files *Input.java* and *example.txt*, the following output line is displayed:

```
The computed sum is 308.5
```

This example demonstrates the use of the *Input* constructor and of the methods *readFloat* and *fails*. The complete class *Input*, listed below, shows that it contains some other useful methods as well:

```
// Input.java: A class to read numbers and characters from text
files.
// Methods of this class, available for other program files:
//   Input(fileName)  (constructor; open input file)
//   Input()          (constructor; prepare for input from keyboard)
//   readInt()        (read an integer)
//   readFloat()      (read a float number)
//   readChar()       (read a character)
//   readString()     (read a string between double quotes)
//   skipRest()       (skip all remaining characters of current line)
//   fails()          (input operation failed)
//   eof()            (failure because of end of file)
//   clear()          (reset error flag)
//   close()          (close input file)
//   pushBack(ch)     (push character ch back into the input stream)
import java.io.*;

class Input {
   private PushbackInputStream pbis;
   private boolean ok = true;
   private boolean eoFile = false;

   Input() {pbis = new PushbackInputStream(System.in);}

   Input(String fileName) {
      try {
         InputStream is = new FileInputStream(fileName);
         pbis = new PushbackInputStream(is);
      } catch (Exception ioe) {ok = false;}
   }

   int readInt() {
      boolean neg = false;
      char ch;
      do {ch = readChar();} while (Character.isWhitespace(ch));
      if (ch == '-') {neg = true; ch = readChar();}
      if (!Character.isDigit(ch)) {
         pushBack(ch);
         ok = false;
         return 0;
      }
      int x = ch - '0';
      for (;;) {
         ch = readChar();
         if (!Character.isDigit(ch)) {
```

```
            pushBack(ch);
            break;
         }
         x = 10 * x + (ch - '0');
      }
      return (neg ? -x : x);
   }

   float readFloat() {
      char ch;
      int nDec = -1;
      boolean neg = false;
      do {ch = readChar();} while (Character.isWhitespace(ch));
      if (ch == '-') {
         neg = true;
         ch = readChar();
      }
      if (ch == '.') {
         nDec = 1;
         ch = readChar();
      }
      if (!Character.isDigit(ch)) {
         ok = false;
         pushBack(ch);
         return 0;
      }
      float x = ch - '0';
      for (;;) {
         ch = readChar();
         if (Character.isDigit(ch)) {
            x = 10 * x + (ch - '0');
            if (nDec >= 0) nDec++;
         }
         else if (ch == '.' && nDec == -1)
            nDec = 0;
         else
            break;
      }
      while (nDec > 0) {x *= 0.1; nDec--;}
      if (ch == 'e' || ch == 'E') {
         int exp = readInt();
         if (!fails()) {
            while (exp < 0) {x *= 0.1; exp++;}
            while (exp > 0) {x *= 10; exp--;}
         }
      }
```

```
      else
         pushBack(ch);
      return (neg ? -x : x);
   }

   char readChar() {
      int ch = 0;
      try {
         ch = pbis.read();
         if (ch == -1) {eoFile = true; ok = false;}
      }
      catch (Exception ioe) {ok = false;}
      return (char) ch;
   }

   String readString() { // Read first string between quotes (").
      String str = "";
      char ch;
      do ch = readChar(); while (!(eof() || ch == '"'));
                                              // Initial quote
      for (;;) {
         ch = readChar();
         if (eof() || ch == '"') // Final quote (end of string)
            break;
         str += ch;
      }
      return str;
   }

   void skipRest() { // Skip rest of line
      char ch;
      do ch = readChar(); while (!(eof() || ch == '\n'));
   }

   boolean fails() {return !ok;}

   boolean eof() {return eoFile;}

   void clear() {
      ok = true;
   }

   void close() {
      if (pbis != null)
         try {pbis.close();} catch (Exception ioe) {ok = false;}
   }
```

```
    void pushBack(char ch) {
        try {pbis.unread(ch);} catch (Exception ioe) {ok = false;}
    }
}
```

Using a call to *readChar* immediately after using *readInt* or *readFloat* causes the character immediately after the number to be read. To realize this, we use the standard Java class *PushbackInputStream*, which enables us to push back, or 'unread' the last character that we have read and that does not belong to the number we are reading.

After an attempt to read a number by using *readInt* or *readFloat*, we can call the *fails* method to check whether that attempt was successful. If *fails* returns *true*, a nonnumeric character, such as a period in the second part of our input files, may have been read. It is then still possible to read that character by using *readChar*. The *clear* method resets the error flag, so that we can resume input, using *fails* again. The method *fails* also returns *true* if an input operation fails because the end of the file is encountered during a call to one of the methods *readInt*, *readFloat* and *readChar*. In that case, the *eof* method also returns *true*. We can use the method *readString* to read a string surrounded by double quotes ("), as we will do in Chap. 8. To skip all remaining characters of the current input line, we use the method *skipRest*.

Obj*3D*: *A Class to Store 3D Objects*

Let us now discuss how the above class *Input* is used to read input files in the method *readObject* of the class *Obj3D*. We will discuss a simplified version of this method. In this fragment each sequence of three dots (...) denotes code that is irrelevant in this discussion because it does not perform any input operations:

```
private boolean readObject(Input inp) {
    for (;;) {
        int i = inp.readInt();
        if (inp.fails()){inp.clear(); break;}
        ...
        float x = inp.readFloat(),
              y = inp.readFloat(),
              z = inp.readFloat();
        addVertex(i, x, y, z);
    }
    ...
    do {  // Skip the line "Faces:"
        ch = inp.readChar(); count++;
    }  while (!inp.eof() && ch != '\n');
```

```
   ...
   // Build polygon list:
   for (;;) {
      Vector vnrs = new Vector();
      for (;;) {
        int i = inp.readInt();
         if (inp.fails()){inp.clear(); break;}
         ...
         vnrs.addElement(new Integer(i));
      }
      ch = inp.readChar();
      if (ch != '.') break;
      // Ignore input lines with only one vertex number:
      if (vnrs.size() >= 2)
         polyList.addElement(new Polygon3D(vnrs));
   }
   inp.close();
   return true;
}
```

We will use the class *Obj3D* to store all data of 3D objects, along with their 2D representations, in such a way that this data is easy to use in our programs. As for the vertices, we store three representations of them:

'Vector' *w* of *Point*3*D* elements:
world coordinates
Array *e* of *Point*3*D* elements:
eye coordinates
Array *vScr* of *Point*2*D* elements:
screen coordinates

Recall that the classes *Point*2*D* and *Point*3*D* were discussed in Sects. 1.4 and 3.9. Since we read the world coordinates from an input file without knowing in advance how many vertices there will be, we use a *Vector* (in the sense of Java) for them. This is different with the eye and screen coordinates. Since we compute these ourselves after reading the world coordinates of all vertices, we know the size of the arrays *e* and *vScr* for them, so that we can allocate memory for them. We use the vertex numbers to indicate the positions in *w*, *e*, and *vScr*. In other words, with an input line of the form

$$i \quad x \quad y \quad z$$

we can find these world coordinates *x*, *y* and *z* of vertex *i* in the *Point*3*D* object

```
(Point3D)w.elementAt(i)
```

We use the *Obj3D* method *eyeAndScreen* to compute the corresponding eye coordinates and store them in the *Point3D* object

```
e[i]
```

This method also computes the corresponding screen coordinates and stores them in the *Point2D* object

```
vScr[i]
```

It follows that *w.size*(), *e.length* and *vScr.length* will be one higher than the highest vertex number that is in use.

We will use the methods *getE*() and *getVScr*() for access to the arrays *e* and *vScr*. The *Vector* object *w* will not be used at all outside the class *Obj3D*.

Another useful method of *Obj3D* is *planeCoeff*. For each face (or polygon) of the object, it computes the coefficients a, b, c and h of the equation

$$ax + by + cz = h \tag{5.10}$$

which describes the plane in which this face lies. Using the first three vertices A, B and C of a polygon, we compute the normal vector $\mathbf{n} = (a, b, c)$ of the plane as the vector product $\mathbf{AB} \times \mathbf{AC}$ (see Sect. 2.2), which we scale such that

$$a^2 + b^2 + c^2 = 1$$

Using the inner product (see Sect. 2.2) of $\mathbf{n}$ and a vector $\mathbf{x} = \mathbf{EP}$ for any point P in the plane, we can write (5.10) as

$$\mathbf{n} \cdot \mathbf{x} = h$$

in which h is positive if the sequence A, B and C is clockwise, that is, if ABC is a back face. On the other hand, if ABC is not a back face, h is negative and the sequence A, B and C is counter-clockwise. Recall that the positive z-axis (in the eye-coordinate system) points towards us, as is more or less the case with the normal vector $\mathbf{n}$ of a visible face. However, the vector $\mathbf{x}$ points the other way, from E to P, which implies that the inner product $\mathbf{n} \cdot \mathbf{x} = h$ will be negative for a visible face and positive for a back face. The absolute value of h is the distance between the eye E and the plane in question (see Exercise 6.5). We will use the coefficients a, b, c and h on several occasions. In the class *Obj3D*, the method *planeCoeff* computes these coefficients, after which they are stored in *Polygon3D* objects, as we will see shortly.

Since the file *Obj3D.java* is considerably larger than the program files we have seen so far, it is not listed here but you can find it in Appendix B. Here is a summary of all methods of this class that we can use outside it:

```
boolean read(String fName)// Reads a 3D object file, if possible.
Vector getPolyList()       // Returns polyList, the list of faces.
String getFName()          // File name of current object.
Point3D[] getE()           // Eye coordinates e of vertices.
Point2D[] getVScr()        // Screen coordinates vScr of vertices.
Point2D getImgCenter()     // Center of image in screen coordinates.
float getRho()             // Rho, the viewing distance.
float getD()               // d, scaling factor, also screen distance.
float eyeAndScreen         // Computes eye and screen coordinates and
                           // returns maximum screen-coordinate range.
void planeCoeff()          // Computes the coefficients a, b, c, h
                           // for all faces.
boolean vp(Canvas cv, float dTheta, float dPhi, float fRho)
                           // Changes the viewpoint.
int colorCode(double a, double b, double c)
                           // Computes the color code of a face.
```

We will discuss the method *colorCode* in detail in Sect. 7.7. The public *Obj3D* method *read* calls the private method *readObject*, discussed above, as the following fragment shows:

```
boolean read(String fName) {
   Input inp = new Input(fName);
   ...
   return readObject(inp); // Read from inp into obj
}
```

As we have seen in the simplified version of *readObject*, this method starts by repeatedly reading four numbers, *i*, *x*, *y* and *z*, and calls the method *addVertex*, which keeps track of the minimum and maximum values of *x*, *y* and *z*. The loop in which this reading of four numbers takes place terminates when an attempt to read a vertex number *i* fails because of the word *Faces*. The minimum and maximum coordinate values just mentioned are required for a call to the private method *shiftToOrigin*, which reduces all world-coordinates such that the origin of the coordinate center will coincide with the center of the bounding box of the object.

After skipping the rest of the line on which we encounter the word *Faces*, we enter a loop to read vertex-number sequences representing polygons. Vertex numbers in these sequences may be preceded by a minus sign, as discussed in the previous section.

Since we will often refer to the class *Obj3D*, let us have a look at a simplified version of it:

```
// Obj3D.java: A 3D object and its 2D representation.
// Uses: Point2D (Section 1.4), Point3D (Section 3.9),
//       Polygon3D, Input (Section 5.6).
```

```
class Obj3D {
   ...
   private Vector w = new Vector();         // World coordinates
   private Point3D[] e;                     // Eye coordinates
   private Point2D[] vScr;                  // Screen coordinates
   private Vector polyList = new Vector(); // Polygon3D objects
   ...
   private void addVertex(int i, float x, float y, float z) {
      ...
   }
   ...
}
```

Recall that the complete version is listed in Appendix B.

Tria*: A Class to Store Triangles by Their Vertex Numbers*

We will often store large sets of triangles, the vertices of which have numbers in the same way as letters are normally used in geometry. Since several triangles may share some vertices, it would not be efficient to store the coordinates of each vertex separately for every triangle. For example, suppose in 3D space we have a triangle with vertices 1, 2 and 3 and another triangle with vertices 1, 4 and 2. It will then be efficient to set up a table with the x-, y- and z-coordinates of the four vertices 1, 2, 3 and 4, and denote the triangles only by their vertex numbers. The following class is used to represent triangles in this way:

```
// Tria.java: Triangle represented by its vertex numbers.
class Tria {
   int iA, iB, iC;
   Tria(int i, int j, int k){iA = i; iB = j; iC = k;}
}
```

If there had been no 3D programs in this book other than that for wireframe models, as discussed in Sect. 5.7, it would not have been necessary to store polygons, let alone triangles, as we are now discussing. We could then have restricted ourselves to the edges of 3D objects, that is, to line segments. We will nevertheless store the faces of the objects in this chapter, to prepare for some more interesting programs that do not display hidden lines and faces.

Polygon3D*: A Class to Store 3D Polygons*

Almost at the end of the *Obj3D* method *readObject* you may have noticed the following statement:

```
polyList.addElement(new Polygon3D(vnrs));
```

Here a new *Polygon3D* object is created in which the vertex numbers of a polygon are stored. This object is then added to the *Vector* object *polyList*, a private variable of the class *Obj3D*. The class *Polygon3D* contains a number of methods that we will not use in this chapter. These are related to triangles resulting from polygons and they will be useful in the next chapter, in which we will be dealing with algorithms to eliminate hidden lines and faces. Thanks to the above class *Tria*, the coordinates of the vertices of each triangle are not duplicated. Since these coordinates are stored in arrays that are data members of the *Obj3D* class, we only need to store the numbers *iA*, *iB* and *iC* of the vertices here. Although, in Sect. 2.6, we have already used a method *triangulate* to divide a polygon into triangles, we will need a slightly different one now that we are representing vertices by numbers referring to the *Obj3D* class. This special method for triangulation is part of the class *Polygon3D*, listed below:

```
// Polygon3D.java: Polygon in 3D, represented by vertex numbers
// referring to coordinates stored in an Obj3D object.
// Uses: Point2D (Section 1.4), Tria, Obj3D( Section 5.6).
import java.util.*;

class Polygon3D {
   private int[] nrs;
   private double a, b, c, h;
   private Tria[] t;

   Polygon3D(Vector<Integer> vnrs) {
      int n = vnrs.size();
      nrs = new int[n];
      for (int i = 0; i < n; i++)
         nrs[i] = ((Integer) vnrs.elementAt(i)).intValue();
   }

   int[] getNrs() {return nrs;}
   double getA() {return a;}
   double getB() {return b;}
   double getC() {return c;}
   double getH() {return h;}
```

```
   void setAbch(double a, double b, double c, double h) {
      this.a = a; this.b = b; this.c = c; this.h = h;
   }

   Tria[] getT() {return t;}

   Tria[] triangulate(Obj3D obj) {
      // Successive vertex numbers (CCW) in vector nrs.
      // Resulting triangles will be put in array t.
      Point2D[] vScr = obj.getVScr();
      Polygon2D polygon = new Polygon2D(vScr, nrs);
      t = polygon.triangulate();
      return t;
   }
}
```

The vertex numbers of the given polygon are available in the array *nrs*, while those of each resulting triangle are stored in an element of the array *t*. The method *triangulate* of this class *Polygon*3*D* is rather simple because we can benefit from the method *triangulate* of the class *Polygon*2*D*, discussed in Sect. 2.6.

Canvas3D*: An Abstract Class to Adapt the Java Class* **Canvas**

The canvas classes we will be using will contain the methods *getObj* and *setObj*, to retrieve and store a reference to an *Obj*3*D* object. In view of a separate frame class, *Fr*3*D*, we need to define the following abstract class:

```
// Canvas3D.java: Abstract class.
import java.awt.*;
abstract class Canvas3D extends Canvas {
   abstract Obj3D getObj();
   abstract void setObj(Obj3D obj);
   boolean specularDesired; // Not used until Chapter 6
}
```

Remember, abstract classes are only useful to create subclasses, not to define objects. Any (non-abstract) subclass of *Canvas*3*D* is simply a subclass of the standard class *Canvas*, except that it is guaranteed to define the methods *getObj* and *setObj*. For example, in the next section we will discuss a class *CvWireframe* of which the first line reads.

```
class CvWireframe extends Canvas3D
```

By writing here *Canvas3D* instead of *Canvas*, we are obliged to define the methods *getObj* and *setObj* in this *CvWireframe* class, and in return we are allowed to call these two methods for any object of class *CvWireframe*. We will clarify the use of the abstract class *Canvas3D* further at the end of this section.

Fr3D: *A Frame Class for 3D Programs*

The class *Fr3D* will be used in four 3D programs of this book, *Wireframe.java* (Sect. 5.7), *HLines.java* (Sect. 6.1), *Painter.java* (Sect. 6.3) and *ZBuf.java* (Sect. 6.4). Since these programs will have the same menus, it makes sense to let them share the file *Fr3D.java*, listed below, in which much of the code is related to these menus. The Java compiler accepts the calls *cv.getObj*() and *cv.setObj*(*obj*) in this file because, as *Canvas3D* is an abstract class, the actual type of *cv* can only be a subclass of it. As we have just seen, this implies that this subclass will define the methods *getObj* and *setObj*:

```
// Fr3D.java: Frame class to deal with menu commands and other
//    user actions.
import java.awt.*;
import java.awt.event.*;

class Fr3D extends Frame implements ActionListener {
   protected MenuItem open, exit, eyeUp, eyeDown, eyeLeft, eyeRight,
         incrDist, decrDist, phongChoice;
   protected String sDir;
   protected Canvas3D cv;
   protected Menu mF, mV;
   private MenuItem exportHPGL;
   private Boolean hiddenLines, lineDrawing;

   Fr3D(String argFileName, Canvas3D cv, String textTitle) {
      super(textTitle);
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      this.cv = cv;
      MenuBar mBar = new MenuBar();
      setMenuBar(mBar);
      mF = new Menu("File"); mV = new Menu("View");
      mBar.add(mF); mBar.add(mV);
      hiddenLines = cv instanceof CvHLines;
      lineDrawing = hiddenLines || cv instanceof CvWireframe;
```

```
open = new MenuItem("Open", new MenuShortcut(KeyEvent.VK_O));
eyeDown = new MenuItem("Viewpoint Down",
                       new MenuShortcut(KeyEvent.VK_DOWN));
eyeUp = new MenuItem("Viewpoint Up",
                     new MenuShortcut(KeyEvent.VK_UP));
eyeLeft = new MenuItem("Viewpoint to Left",
                       new MenuShortcut(KeyEvent.VK_LEFT));
eyeRight = new MenuItem("Viewpoint to Right",
                        new MenuShortcut(KeyEvent.VK_RIGHT));

incrDist = new MenuItem("Increase viewing distance",
                        new MenuShortcut(KeyEvent.VK_INSERT));
decrDist = new MenuItem("Decrease viewing distance",
                        new MenuShortcut(KeyEvent.VK_DELETE));
exit = new MenuItem("Exit", new MenuShortcut(KeyEvent.VK_Q));
mF.add(open); mF.add(exit);
mV.add(eyeDown); mV.add(eyeUp);
mV.add(eyeLeft); mV.add(eyeRight);
mV.add(incrDist); mV.add(decrDist);

open.addActionListener(this);
exit.addActionListener(this);
eyeDown.addActionListener(this);
eyeUp.addActionListener(this);
eyeLeft.addActionListener(this);
eyeRight.addActionListener(this);
incrDist.addActionListener(this);
decrDist.addActionListener(this);

if (hiddenLines){
   exportHPGL = new MenuItem("Export HP-GL");
   mF.add(exportHPGL);     // mF defined in Fr3D
   exportHPGL.addActionListener(this);
}
else
if (!lineDrawing) {
   phongChoice = new MenuItem(
   (cv.specularDesired?"Diffuse":"Specular")+" illumination");
   mV.add(phongChoice);
   phongChoice.addActionListener(this);
}

add("Center", cv);
Dimension dim = getToolkit().getScreenSize();
setSize(dim.width / 2, dim.height / 2);
```

```
      setLocation(dim.width / 4, dim.height / 4);
      if (argFileName != null) {
         Obj3D obj = new Obj3D();
         if (obj.read(argFileName)) {cv.setObj(obj); cv.repaint();}
         else {
            System.out.println("Cannot open input file " + argFileName);
         }

      }
      cv.setBackground(new Color(180, 180, 255));
      setVisible(true);
   }

   void vp(float dTheta, float dPhi, float fRho) {// Viewpoint
      Obj3D obj = cv.getObj();
      if (obj == null || !obj.vp(cv, dTheta, dPhi, fRho))
         Toolkit.getDefaultToolkit().beep();
   }

   public void actionPerformed(ActionEvent ae) {
      if (ae.getSource() instanceof MenuItem) {
         MenuItem mi = (MenuItem) ae.getSource();
         if (mi == open) {
            FileDialog fDia = new FileDialog(Fr3D.this, "Open",
                  FileDialog.LOAD);
            fDia.setDirectory(sDir);
            fDia.setFile("*.dat");
            fDia.setVisible(true);
            String sDir1 = fDia.getDirectory();
            String sFile = fDia.getFile();
            String fName = sDir1 + sFile;
            Obj3D obj = new Obj3D();
            if (obj.read(fName)) {
               sDir = sDir1;
               cv.setObj(obj);
               cv.repaint();
            }
         }
         else {
            if (mi == exit) System.exit(0); else
            if (mi == eyeDown) vp(0, .1F, 1); else
            if (mi == eyeUp) vp(0, -.1F, 1); else
            if (mi == eyeLeft) vp(-.1F, 0, 1); else
            if (mi == eyeRight) vp(.1F, 0, 1); else
            if (mi == incrDist) vp(0, 0, 2); else
```

```
            if (mi == decrDist) vp(0, 0, .5F); else
            if (mi == phongChoice) {
                cv.specularDesired = !cv.specularDesired;
                phongChoice.setLabel(
            (cv.specularDesired?"Diffuse":"Specular")+"illumination");
             cv.repaint();
            }
            else
            if (mi == exportHPGL) {
               Obj3D obj = cv.getObj();
               if (obj != null) {
                  ((CvHLines)cv).setHPGL(new HPGL(obj));
                  cv.repaint();
               }
            }
         }
      }
   }
}
```

Some menu items and associated program code will not be used in this chapter, but they are already inserted here so in the next chapter we need not update this *Fr3D* class. Notice the use of the abstract class *Canvas*3*D* in the line.

```
Fr3D(String argFileName, Canvas3D cv, String textTitle)
```

almost at the beginning of class *Fr*3*D*. We cannot replace *Canvas*3*D* with *Canvas* here because then the compiler would not accept the statement

```
cv.setObj(obj);
```

in the if-statement near the end of the *Fr*3*D* constructor. After all, *setObj* is not a method of the standard Java class *Canvas*. Since, in the next section we will actually be using the class *CvWireframe*, of which *setObj* is really a method, it is tempting to write *CvWireframe* instead of Canvas3D. This would indeed work, but then we would not be able to use the class *Fr*3*D* also in programs with canvas classes other than *CvWireframe*, as we will do in the next chapter. We now see that the abstract class *Canvas*3*D* is very useful. It is general enough to be used in several of our programs and yet less general than the standard *Canvas* class in that it 'promises' an implementation of the methods *setObj* and *getObj*. We will discuss some aspects of the class *Fr*3*D* in the next chapter, when we will be using it.

5.7 A Program for Wireframe Models

It is now time to see the classes of the previous section in action. A relatively simple way of displaying 3D objects, bounded by polygons, is by drawing all the edges of these polygons, as we did for a cube in Sect. 5.4. The classes we have just seen enable us to write a general program *Wireframe.java*, which can read input files by using a menu command *Open* and which enable the user to view the object from any reasonable viewpoint, by using either menu commands or the keyboard. The comment lists all classes that *Wireframe.java*, directly or indirectly, uses. Since the program does not hide object edges that, for opaque objects, are invisible, its practical usefulness will be limited. It is nevertheless presented here as an introduction, demonstrating many aspects that will recur in more practical but also more sophisticated programs in the next chapter. As you can see both in program file *Fr3D.java* of the previous section and in Fig. 5.13, there will be two menus, *File* and *View*. There are two menu items, *Open* and *Exit*, in the *File* menu, and six in the *View* menu. The latter enables the user to change the viewpoint, as shown in Fig. 5.13. These eight menu items will be available in all four 3D programs (*Wireframe*, *HLines*, *Painter* and *ZBuf*) discussed in this book. In each, an object of the frame class *Fr3D*, listed in the previous section, is created in the *main* method as shown for the class *Wireframe* below.

```
// Wireframe.java: Perspective drawing using an input file that lists
//    vertices and faces.
// Uses: Point2D (Section 1.4), Tools2D (Section 2.3),
//    Point3D (Section 3.9).
//    Input, Obj3D, Tria, Polygon3D, Canvas3D, Fr3D (Section 5.6),
//    CvWireframe (Section 5.7).
import java.awt.*;

public class Wireframe extends Frame {
   public static void main(String[] args) {
      new Fr3D(args.length > 0 ? args[0] : null, new CvWireframe(),
            "Wire-frame model");
   }
}
```

This rather simple program file accepts an optional program argument, which may be supplied to specify the name of the input file. This is what the first argument, a conditional expression, of the *Fr3D* constructor is about. The second argument of this constructor generates an object of class *CvWireframe*, which does almost all the work. Finally, the third argument specifies the text, here *Wire-frame model*, that we want to appear in the title bar of the window. The class *CvWireframe* is listed below:

```java
// CvWireframe.java: Canvas class for class Wireframe.
import java.awt.*;
import java.util.*;

class CvWireframe extends Canvas3D {
   private int maxX, maxY, centerX, centerY;
   private Obj3D obj;
   private Point2D imgCenter;

   Obj3D getObj() {return obj;}
   void setObj(Obj3D obj) {this.obj = obj;}
   int iX(float x) {return Math.round(centerX + x - imgCenter.x);}
   int iY(float y) {return Math.round(centerY - y + imgCenter.y);}

   public void paint(Graphics g) {
      if (obj == null) return;
      Vector<Polygon3D> polyList = obj.getPolyList();
      if (polyList == null) return;
      int nFaces = polyList.size();
      if (nFaces == 0) return;

      Dimension dim = getSize();
      maxX = dim.width - 1; maxY = dim.height - 1;
      centerX = maxX / 2; centerY = maxY / 2;
      // ze-axis towards eye, so ze-coordinates of
      // object points are all negative.
      // obj is a java object that contains all data:
      // - Vector w (world coordinates)
      // - Array e (eye coordinates)
      // - Array vScr (screen coordinates)
      // - Vector polyList (Polygon3D objects)

      // Every Polygon3D value contains:
      // - Array 'nrs' for vertex numbers
      // - Values a, b, c, h for the plane ax+by+cz=h.
      // (- Array t (with nrs.length-2 elements of type Tria))

      obj.eyeAndScreen(dim);
      // Computation of eye and screen coordinates.

      imgCenter = obj.getImgCenter();
      obj.planeCoeff(); // Compute a, b, c and h.
      Point2D[] vScr = obj.getVScr();

      g.setColor(Color.black);
```

```
      for (int j = 0; j < nFaces; j++) {
         Polygon3D pol = polyList.elementAt(j);
         int nrs[] = pol.getNrs();
         for (int iA = 0; iA < nrs.length; iA++) {
            int iB = (iA + 1) % nrs.length;
            int na = Math.abs(nrs[iA]), nb = nrs[iB];
           // abs in view of minus signs discussed in Section 5.5.
            if (nb >= 0) {
               Point2D a = vScr[na], b = vScr[nb];
               g.drawLine(iX(a.x), iY(a.y), iX(b.x), iY(b.y));
            }
         }
      }
   }
}
```

Thanks to the other classes discussed in the previous section, this program file is rather small. However, due to the object-oriented character of Java, the flow of control of the whole program may not be immediately clear. In particular, you may wonder how starting the program leads to reading a 3D object file and displaying the desired image. Let us begin with the *main* method in the file *Writeframe.java*. Here an *Fr3D* object is created by calling its constructor, and a *CvWireframe* object is created at the same time in the second argument of this *Fr3D* constructor call. As we have seen in the previous section, the first line of class *Fr3D* reads

```
class Fr3D extends Frame implements ActionListener {
```

which indicates that this class contains a method *actionPerformed*. This method is called when a menu command is given. In particular, the use of the *Open* command in the *File* menu gives rise to the execution of a fragment that causes a standard dialog box for ‘Open file’ to appear, as you can see in the method *actionPerformed*. This fragment contains the following:

```
Obj3D obj = new Obj3D();
if (obj.read(fName)) {
   sDir = sDir1;
   cv.setObj(obj);
   cv.repaint();
}
```

Here we see that the *Obj3D* method *read* is called, with the file name supplied by the user supplied as an argument, so that this method can read any 3D object provided by the user. We also find here a call to *setObj*, a method defined in the above class *CvWireframe* as

```
void setObj(Obj3D obj){this.obj = obj;}
```

As a result, the *CvWireframe* class gets access to the *Obj3D* object that contains all data for the real 3D object. The above if-statement also contains a call to the standard Java *Canvas* method *repaint*, which causes the method *paint* of our *CvWireframe* class to be called. The actual computation and display of the image is done in this method *paint*, although much of the computation work is delegated to methods of other classes. An example of this is the statement

```
obj.eyeAndScreen(dim);
```

which calls the *Obj3D* method *eyeAndScreen* to compute the eye and screen coordinates of the object.

A Demonstration

It is now time to see the program *Wireframe.java* in action. For example, let us use the object with vertices 1, 2, ..., 6, shown in Fig. 5.18.

With dimensions 1, 3 and 5 for, respectively, the thickness, the width and the height of the object, the following data file specifies it:

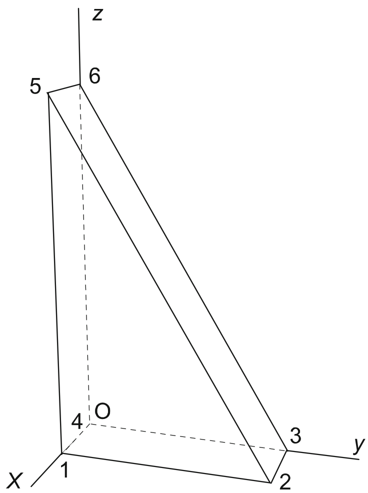


Fig. 5.18 Object and vertex numbers

```
1    1 0 0
2    1 3 0
3    0 3 0
4    0 0 0
5    1 0 5
6    0 0 5
Faces:
1 2 5.
3 4 6.
2 3 6 5.
1 5 6 4.
1 4 3 2.
```

Running program *Wireframe.java* and opening the above input file produces a window that shows a default 3D view of the object defined by this input file. Figure 5.19 shows the window after the user has clicked on the *View* menu.

By moving the viewpoint, using the *Viewpoint to Right* command from the *View* menu or its shortcut *Ctrl* + *Arrow right*, a number of times we obtain a different view of the same object, as shown in Fig. 5.20.

Note that the program chooses a default viewing distance which is quite reasonable. If desired we can increase or decrease this distance using the last two commands from the *View* menu.

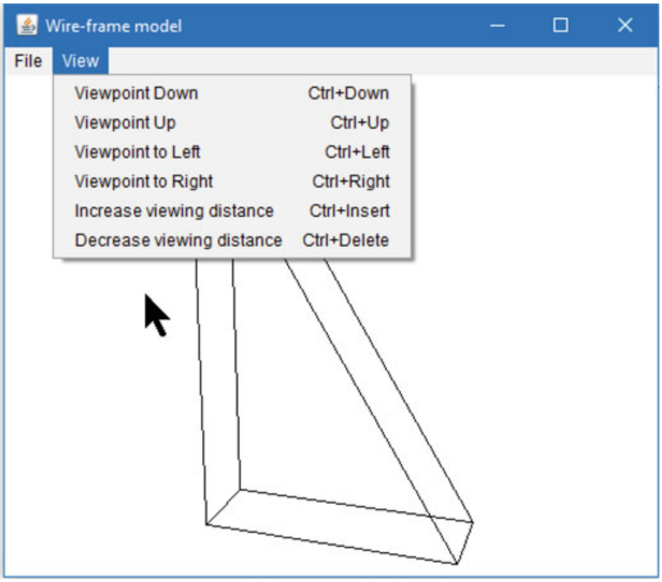


Fig. 5.19 Default view of object

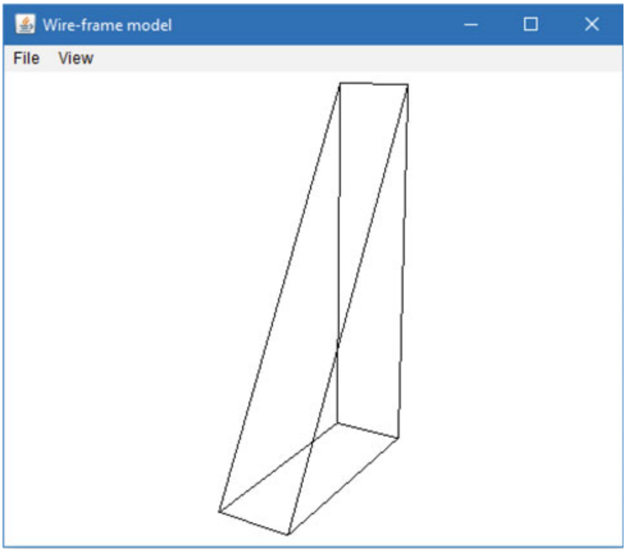


Fig. 5.20 Same object viewed from the back

5.8 Automatic Generation of Object Specification

As long as 3D objects do not have too many vertices and the vertex coordinates are easily available, it is not difficult to create 3D specifications as input files by entering all data manually, using a text editor. This is the case, for example, with the solid letter A, as previously discussed. If there are many vertices, which is normally the case if we approximate curved surfaces, we had better generate 3D data files by special programs. This section explains how to automatically generate 3D specifications through an example. The generated specification files are accepted not only by the program *Wireframe.java* of the previous section, but also by the programs *HLines.java,Painter.java*, *ZBuf.java*, discussed in Chap. 6. Most illustrations in this chapter have been obtained by using *HLines.java*.

Many 3D objects are bounded by curved surfaces. We can approximate these by a set of polygons. An example is a hollow cylinder as shown in Fig. 5.21 on the right. Both representations of hollow cylinders (or rather, hollow prisms) of this figure were obtained by running the program *Cylinder.java*, which we will be discussing, followed by the execution of program *HLines.java*. Although the object shown on the left in Fig. 5.21 is a (hollow) *prism*, not a cylinder, we will consistently use the term *cylinder* in this discussion.

The user will be able to enter the diameters of both the (outer) cylinder and the (inner) cylindrical hole. If the latter, smaller diameter is zero, our program will produce a solid cylinder instead of a hollow one. For simplicity, we will ignore this

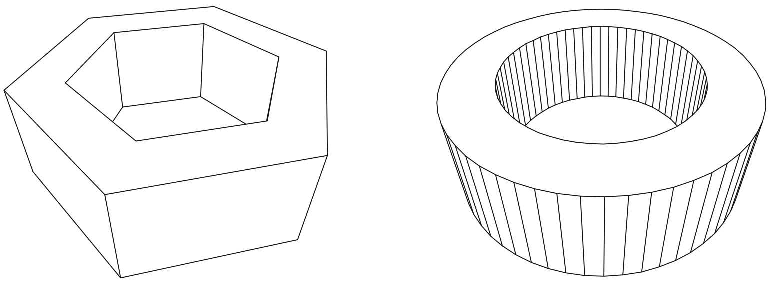


Fig. 5.21 Hollow cylinders with $n = 6$ (*left*) and $n = 60$ (*right*)

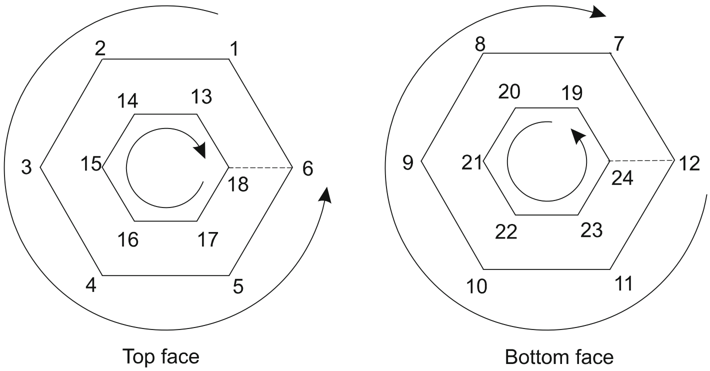


Fig. 5.22 Vertex numbering

special case, with only half the number of vertices, in our discussion below, but simply implement it in the program.

For some sufficiently large integer n (not less than 3), we choose n equidistant points on the outer circle (with radius R) of the top face, and we choose n similar points on the bottom face. Then we approximate the outer cylinder by a prism whose vertices are these $2n$ points. The inner circle (of the cylindrical hole) has radius r ($<R$). The hollow cylinder has height h. Let us use the z-axis of our coordinate system as the cylinder axis. The cylindrical hole is approximated by rectangles in the same way as the outer cylinder. The bottom face lies in the plane $z = 0$ and the top face in the plane $z = h$. A vertex of the bottom face lies on the positive x-axis. Let us set $h = 1$. Then for given values n, R, and r, the object to be drawn and its position are then completely determined. We shall first deal with the case $n = 6$ and generalize this later for arbitrary n. We number the vertices as shown in Fig. 5.22.

For each vertex i of the top face there is a vertical edge that connects it with vertex $i + 6$. We can specify the top face by the following sequence:

```
1 2 3 4 5 6 -18 17 16 15 14 13 18 -6.
```

Here the pairs (6, −18) and (18, −6) denote an artificial edge, as previously discussed. The bottom face on the right is viewed here from the positive z-axis, but in reality only the other side is visible. The orientation of this bottom face is therefore opposite to what we see in Fig. 5.22 on the right, so that we can specify this face as.

```
12 11 10 9 8 7 -19 20 21 22 23 24 19 -7.
```

Since $n = 6$, we have $12 = 2n$, $18 = 3n$ and $24 = 4n$, so the above sequences are special cases of

$$1 \quad 2 \quad \ldots \quad n \quad -3n \quad 3n-1 \quad 3n-2 \quad \ldots \quad 2n+1 \quad 3n \quad -n.$$

and

$$2n \quad 2n-1 \quad \ldots \quad n+1 \quad -(3n+1) \quad 3n+2 \quad 3n+3 \quad \ldots \quad 4n \quad 3n+1 \quad -(n+1).$$

Let us define

$$\delta = \frac{2\pi}{n}$$

Since, in Fig. 5.22, on the left, vertex 6 lies on the positive x-axis and according to geometry in Fig. 5.23 (outer circle), the Cartesian coordinates of the vertices on the top face are as follows:

$$\begin{aligned} x_i &= R \cos i\delta \\ y_i &= R \sin i\delta \quad (i = 1, \ldots, n; \text{outer circle}) \\ z_i &= h \end{aligned}$$

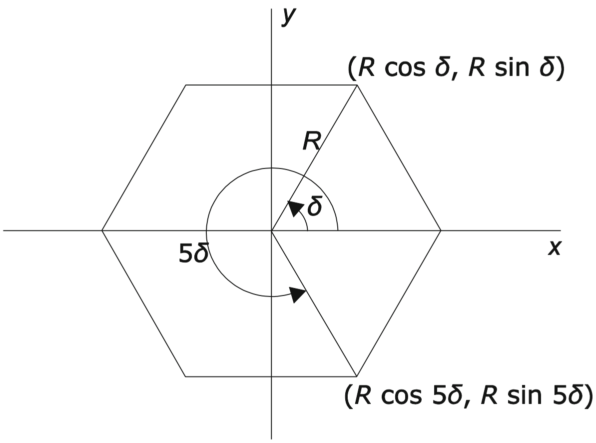


Fig. 5.23 Calculating vertex coordinates

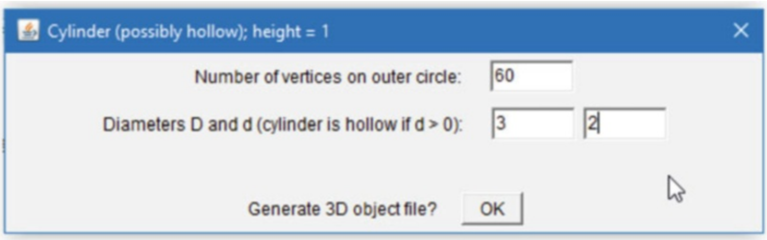


Fig. 5.24 Dialog box for (possibly hollow) cylinder

$$\begin{aligned} x_i &= r\cos(i-2n)\delta \\ y_i &= r\sin(i-2n)\delta \quad (i = 2n+1, \ldots, 3n; \text{inner circle}) \\ z_i &= h \end{aligned}$$

For the bottom face we have

$$\begin{aligned} x_i &= x_{i-n} \\ y_i &= y_{i-n} \quad (i = n+1, \ldots, 2n, 3n+1, \ldots, 4n) \\ z_i &= 0 \end{aligned}$$

A program based on the above analysis can be written in any programming language. Using Java for this purpose, we can choose between an old fashioned, text-line oriented solution and a graphical user interface with, for example, a dialog box with text fields and a button as shown in Fig. 5.24.

This dialog box contains a title bar, and seven so-called *components*: three *labels* (that is, static text in the gray area), three text fields in which the user can enter data, and an OK button. Programming the layout of a dialog box in Java can be done in several ways, none of which is particularly simple. Here we do this by using three *panels*:

- Panel $p1$ at the top, or *North*, for both the label *Number of vertices on outer circle* and a text field in which this number is to be entered.
- Panel $p2$ in the middle, or *Center*, for the label *Diameters D and d (cylinder is hollow if d > 0)* and two text fields for these diameters.
- Panel $p3$ at the bottom, or *South*, for the label *Generate 3D object file?* and an OK button.

Since there are only a few components in each panel, we can use the default *FlowLayout* layout manager for the placements of these components in the panels. By contrast, the panels are placed above one another by using *BorderLayout*, as the above words *North*, *Center* and *South*, used in the program as character strings, indicate. As in many other graphics programs in this book, we use two classes in this program, but this time there is a dialog class instead of a canvas class. Another difference is that we do not display the frame, but restrict the graphical output to the dialog box. (We cannot omit the frame class altogether because the *Dialog* constructor requires a 'parent frame' as an argument.) Recall that we previously used calls to *setSize*, *setLocation* and *show* in the constructor of the frame class. We simply omit these calls to prevent the frame from appearing on the screen. Obviously, we must not omit such calls in the constructor of our dialog class, called *DlgCylinder* in the program. As for the generation of the hollow cylinder itself, as discussed above, this can be found in the method *genCylinder*, which follows this constructor:

```
// Cylinder.java: Generating an input file for a
//                (possibly hollow) cylinder.
import java.awt.*;
import java.awt.event.*;
import java.io.*;

public class Cylinder extends Frame {
   public static void main(String[] args) {new Cylinder();}
   Cylinder() {new DlgCylinder(this);}
}

class DlgCylinder extends Dialog {
   TextField tfN = new TextField(5);
   TextField tfOuterDiam = new TextField(5);
   TextField tfInnerDiam = new TextField(5);
   Button button = new Button("  OK  ");
   FileWriter fw;

   DlgCylinder(Frame fr) {
      super(fr, "Cylinder (possibly hollow); height = 1", true);
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {
            dispose();
            System.exit(0);
         }
      });
      Panel p1 = new Panel(), p2 = new Panel(), p3 = new Panel();
      p1.add(new Label("Number of vertices on outer circle: "));
      p1.add(tfN);
      p2.add(new Label(
           "Diameters D and d (cylinder is hollow if d > 0): "));
      p2.add(tfOuterDiam); p2.add(tfInnerDiam);
      p3.add(new Label("Generate 3D object file?"));
      p3.add(button);
      setLayout(new BorderLayout());
```

```
      add("North", p1);
      add("Center", p2);
      add("South", p3);

      button.addActionListener(new ActionListener() {
         public void actionPerformed(ActionEvent ae) {
            int n = 0;
            float dOuter = 0, dInner = 0;
            try {
               n = Integer.valueOf(tfN.getText()).intValue();
               dOuter =
                Float.valueOf(tfOuterDiam.getText()).floatValue();
               dInner =
                Float.valueOf(tfInnerDiam.getText()).floatValue();
               if (dInner < 0) dInner = 0;
               if (n < 3 || dOuter <= dInner)
                  Toolkit.getDefaultToolkit().beep();
               else {
                  try {
                     genCylinder(n, dOuter / 2, dInner / 2);
                  } catch (IOException ioe) {
                  }
                  dispose();
                  System.exit(0);
               }
            } catch (NumberFormatException nfe) {
               Toolkit.getDefaultToolkit().beep();
            }
         }
      }});
      Dimension dim = getToolkit().getScreenSize();
      setSize(3 * dim.width / 4, dim.height / 4);
      setLocation(dim.width / 8, dim.height / 8);
      setVisible(true);
   }

   void genCylinder(int n, float rOuter, float rInner)
         throws IOException {
      int n2 = 2 * n, n3 = 3 * n, n4 = 4 * n;
      fw = new FileWriter("Cylinder.dat");
      double delta = 2 * Math.PI / n;
      for (int i = 1; i <= n; i++) {
         double alpha = i * delta,
               cosa = Math.cos(alpha), sina = Math.sin(alpha);
```

```
      for (int inner = 0; inner < 2; inner++) {
         double r = (inner == 0 ? rOuter : rInner);
         if (r > 0)
            for (int bottom = 0; bottom < 2; bottom++) {
               int k = (2 * inner + bottom) * n + i;
               // Vertex numbers for i = 1:
               // Top: 1 (outer) 2n+1 (inner)
               // Bottom: n+1 (outer) 3n+1 (inner)
               wi(k); // w = write, i = int, r = real
               wr(r * cosa); wr(r * sina); // x and y
               wi(1 - bottom); // bottom: z = 0; top: z = 1
               fw.write("\r\n");
            }
      }
   }
   fw.write("Faces:\r\n");
   // Top boundary face:
   for (int i = 1; i <= n; i++) wi(i);
   if (rInner > 0) {
      wi(-n3); // Invisible edge
      for (int i = n3 - 1; i >= n2 + 1; i--) wi(i);
      wi(n3); wi(-n); // Invisible edge again.
   }
   fw.write(".\r\n");
   // Bottom boundary face:
   for (int i = n2; i >= n + 1; i--) wi(i);
   if (rInner > 0) {
      wi(-(n3 + 1));
      for (int i = n3 + 2; i <= n4; i++) wi(i);
      wi(n3 + 1); wi(-(n + 1));
   }
   fw.write(".\r\n");
   // Vertical, rectangular faces:
   for (int i = 1; i <= n; i++) {
      int j = i % n + 1;
      // Outer rectangle:
      wi(j); wi(i); wi(i + n); wi(j + n); fw.write(".\r\n");
      if (rInner > 0) { // Inner rectangle:
         wi(i + n2); wi(j + n2); wi(j + n3); wi(i + n3);
         fw.write(".\r\n");
      }
   }
   fw.close();
}
```

```
   void wi(int x) throws IOException {
      fw.write(" " + String.valueOf(x));
   }

   void wr(double x) throws IOException {
      if (Math.abs(x) < 1e-9) x = 0;
      fw.write(" " + String.valueOf((float) x));
      // float instead of double to reduce the file size
   }
}
```

The number 60 entered in the top text field of Fig. 5.24 refers to the hollow cylinder shown in Fig. 5.24 on the right. The hollow prism shown on the left in this figure is obtained by replacing 60 with 6. In that case the following file is generated:

```
 1 0.75 1.299038 1
 7 0.75 1.299038 0
 13 0.5 0.8660254 1
 19 0.5 0.8660254 0
 2 -0.75 1.299038 1
 8 -0.75 1.299038 0
 14 -0.5 0.8660254 1
 20 -0.5 0.8660254 0
 3 -1.5 0.0 1
 9 -1.5 0.0 0
 15 -1.0 0.0 1
 21 -1.0 0.0 0
 4 -0.75 -1.299038 1
 10 -0.75 -1.299038 0
 16 -0.5 -0.8660254 1
 22 -0.5 -0.8660254 0
 5 0.75 -1.299038 1
 11 0.75 -1.299038 0
 17 0.5 -0.8660254 1
 23 0.5 -0.8660254 0
 6 1.5 0.0 1
 12 1.5 0.0 0
 18 1.0 0.0 1
 24 1.0 0.0 0
Faces:
 1 2 3 4 5 6 -18 17 16 15 14 13 18 -6.
 12 11 10 9 8 7 -19 20 21 22 23 24 19 -7.
 2 1 7 8.
 13 14 20 19.
 3 2 8 9.
```

```
14 15 21 20.
4 3 9 10.
15 16 22 21.
5 4 10 11.
16 17 23 22.
6 5 11 12.
17 18 24 23.
1 6 12 7.
18 13 19 24.
```

Recall that we have already discussed the first two lines that follow the word *Faces*. More interesting examples on generating input files for different 3D objects can be found in Appendix D.

Exercises

5.1 Modify program *CubePers.java* of Sect. 5.4 in such a way that, *with the given viewpoint*, only the visible edges are drawn as solid black lines; draw the other, invisible lines in a different color, or, as usually in mechanical engineering, as dashed lines (see Exercise 1.5).

5.2 Use a *fillPolygon* method to display only the top, right and front faces of Fig. 5.10 as filled polygons of different colors.

5.3 To prepare for Exercise 5.5, extend program *CubePers.java* of Sect. 5.4 so that two cubes beside each other are generated.

5.4 Use the class *Rota3D* of Sect. 3.9 to apply *animation with double buffering* to the cube of Sect. 5.4, using a rotation about some line, say, 0–6 (see Fig. 5.11) through some small angle. If you are unfamiliar with animation in Java or with the Java class *Image*, required for double buffering, you will find the program *Anim.java* in Appendix E helpful.

5.5 As Exercise 5.4, but the rotation is to be applied to the two cubes of Exercise 5.3. Use different axes of rotation and increase the rotation angles for the two cubes by different amounts, so that the cubes seem to rotate independently of each other, with different speeds. Figure 5.25 shows a snapshot of the two cubes, each rotating about one of its vertical edges.

In the following exercises you are asked to generate input files as discussed in Sect. 5.5. The illustrations below, however, were obtained using program *HLines* of Chap. 6. You can also try this program to get better graphical results with it than with program *Wireframe* of Sect. 5.7. The same applies to the two hidden-face programs *Painter* and *ZBuf*, also discussed in Chap. 6.

5.6 Write a program *BookView.java* that can generate a data file for an open book. Enable the user to supply the number of sheets, the page width and height, and the name of the output file as program arguments. For example, the files

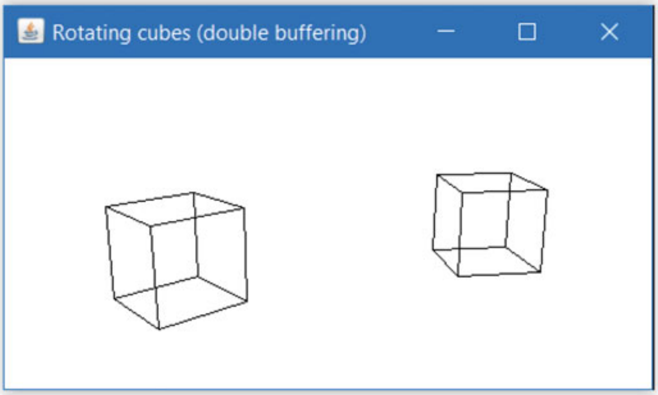


Fig. 5.25 Screenshot of two cubes rotating about vertical axes

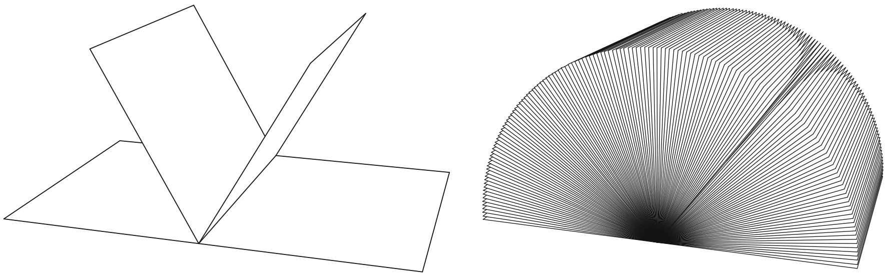


Fig. 5.26 Two open books; numbers of sheets: 4 on the *left* and 150 on the *right*

*bookv*4*.dat* and *bookv*150*.dat* for the books shown in Fig. 5.26 were obtained by executing the following commands:

```
java BookView 4 15 20 bookv4.dat
java BookView 150 15 20 bookv150.dat
```

Apply the program *HLines.java* to it to generate HP-GL files. Import these files in a text processor or drawing program, as was done twice for Fig. 5.26.

5.7 Write a program to generate a globe model of a sphere, as shown in Fig. D.4 of Appendix D. Enable the user to supply n as a program argument.

5.8 Write a program to generate a semi-sphere, as shown in Fig. 5.27.

5.9 Generate a great many cubes that are placed beside, behind and above each other (see Fig. 5.28).

5.10 Generate a data file for two tori (the plural form of torus) as shown in Fig. 5.29.

5.11 Write a program to generate a spiral staircase, as shown in Fig. 5.30.

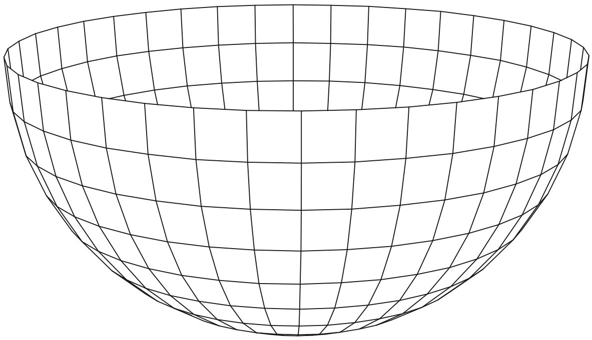


Fig. 5.27 A semi-sphere

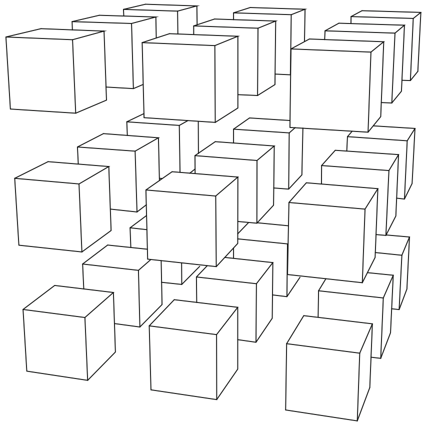


Fig. 5.28 A cube of cubes

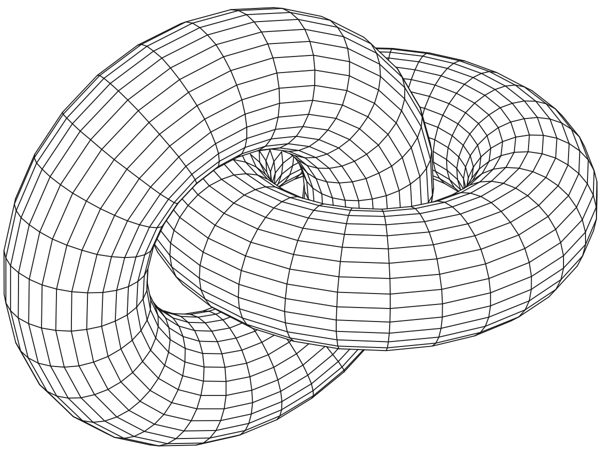
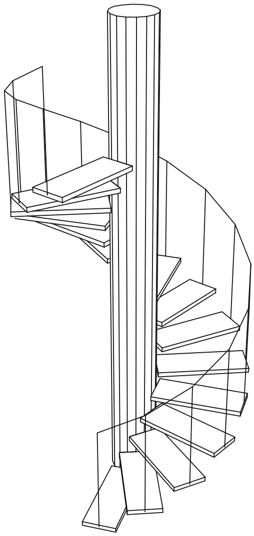


Fig. 5.29 Two tori

Fig. 5.30 A spiral staircase



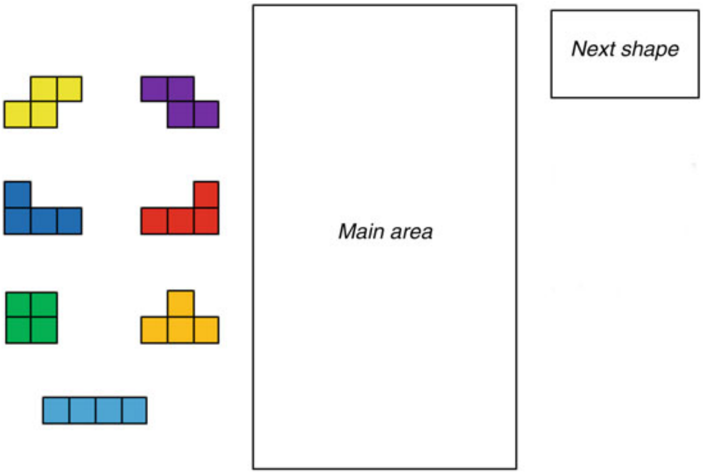


Fig. 5.31 Tetris: seven shapes

5.12 Enhance your adapted Tetris program (from Exercise 1.6 in Chap. 1) to include the following steps and functionality:

- Randomly select one of the seven shapes, as shown in Fig. 5.31, to be displayed in the center at the top of "Main area" and randomly select a different shape to be displayed in "Next shape". The starting orientation of the shape can be fixed.

- The shape at the top of "Main area" moves down (falls) at a constant speed. Once the shape's lowest edge or point touches the bottom of "Main area" or the top edge or a point of another shape, it stops moving and stays there.
- When the cursor is outside of "Main area", each click on the left mouse button moves the falling shape to the left by one square, and similarly, the right button moves the shape to the right by one square. A forward scroll of the mouse wheel will rotate the shape clockwise and a backward scroll will rotate the shape counter-clockwise, both through 90°. If the mouse cursor moves inside "Main area", the falling stops, i.e. a pause.

Chapter 6
Hidden-Line and Hidden-Face Removal

The previous chapter has discussed the specification of 3D objects and their perspective view in preparation for this chapter. We will now first consider line drawings of 3D objects, that engineers usually wish to use. Such line drawings typically display the lines on the back, i.e. invisible lines or *hidden lines*, as dashed lines. Although line drawings might look rather dull compared with colored representations of such objects, there are many technical applications for which they are desired. The first part of this chapter will discuss how to identify hidden lines and omit them in line drawings. The chapter will then describe how backfaces, i.e. invisible faces or *hidden faces*, could be identified by several algorithms. The viewpoint will be taken care of automatically, so that the hidden faces of an object are omitted regardless of the chosen viewpoint. Besides, a face may be only partly visible, for example, if we have two cubes, with the nearer one partly hiding the farther one. The problem of displaying only the visible portions of faces will also be solved in this chapter.

6.1 Hidden-Line Algorithm

Although the faces of a 3D object can be polygons with any number of vertices, we will triangulate these polygons and use the resulting triangles instead. Suppose we are given a set of line segments PQ and a set of triangles ABC in terms of the eye and screen coordinates of all points P, Q, A, B and C. Our task is to draw each line segment PQ as far as none of the triangles ABC obscures them. A line segment (or *line*, for short) may be completely visible, completely invisible or partly visible. For example, in Fig. 6.1, line 1 is completely visible because it lies in front of the triangle on the left and is unrelated to the other two triangles. In contrast, line 2 lies behind the triangle in the middle and is therefore completely invisible. Finally, lines 3 and 4 are not completely visible, but some parts of them are.

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2_6

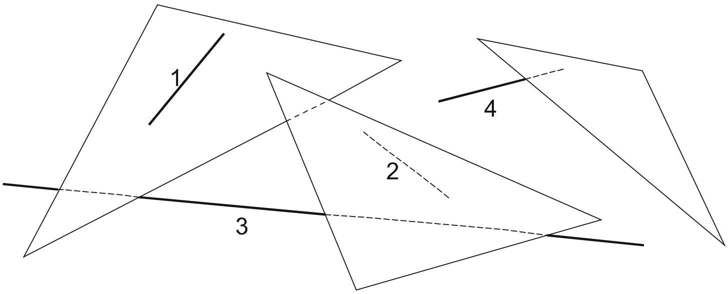
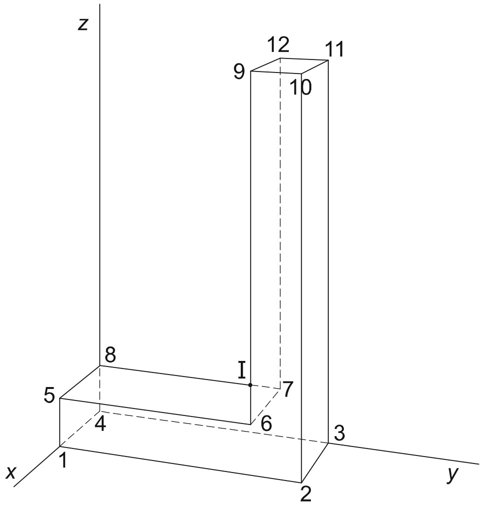


Fig. 6.1 Triangles and line segments

Fig. 6.2 A nonconvex polyhedron



There is another important case: since the edges of the triangles are also line segments, it will frequently occur that a line segment is an edge of the triangle under consideration. Such line segments are to be considered visible, as far as that triangle is concerned. For example, consider the letter L in Fig. 6.2, with the following data file *letterL.dat*:

```
1   20   0   0
2   20  50   0
3    0  50   0
4    0   0   0
5   20   0  10
6   20  40  10
7    0  40  10
8    0   0  10
```

```
 9   20  40  80
10   20  50  80
11    0  50  80
12    0  40  80
Faces:
1   2  10   9   6  5.
3   4   8   7  12 11.
2   3  11  10.
7   6   9  12.
4   1   5   8.
9  10  11  12.
5   6   7   8.
1   4   3   2.
```

Here the only line segment that is partly visible and partly invisible is the line 7-8, which intersects the face 1-2-10-9-6-5, or rather one of the triangles (6-10-9, for example) of which it consists. In the image, the line 7-8 intersects the edge 6-9 of that triangle in point I, which divides this line into the visible part 8-I and the invisible one I-7.

Given the faces of the object, we build a set of triangles and a set of line segments. Then for each line segment PQ, we call a method *lineSegment*, which draws any parts of PQ that are visible. This method is recursive and can be represented in a flowchart in Fig. 6.3, where the numbered tests performed in different steps will be individually discussed in Appendix C.1 and the boxes with double edges denote recursive calls.

This flowchart is equivalent to the following pseudo code, in which I and J are points (between P and Q) on line segment PQ. On the screen we view I and J as intersecting points of PQ with edges of triangle ABC:

```
void lineSegment(line PQ, set s of triangles) {
   Search set s for a triangle ABC that obscures PQ (or part of it)
   If no such triangle found,
      Draw PQ
   Else {
      If triangle ABC leaves part PI of PQ visible
         lineSegment(PI, the remaining triangles of s);   // Recursive call
      If triangle ABC leaves JQ of PQ visible
         lineSegment(JQ, the remaining triangles of s);  // Recursive call
   }
}
```

According to both the flowchart and this pseudo code, the loop that searches the set of triangles terminates as soon as a triangle ABC is found that obscures PQ. If ABC obscures PQ completely, no other action is required. If ABC obscures PQ partly, the parts that are possibly visible are dealt with recursively, using the remaining

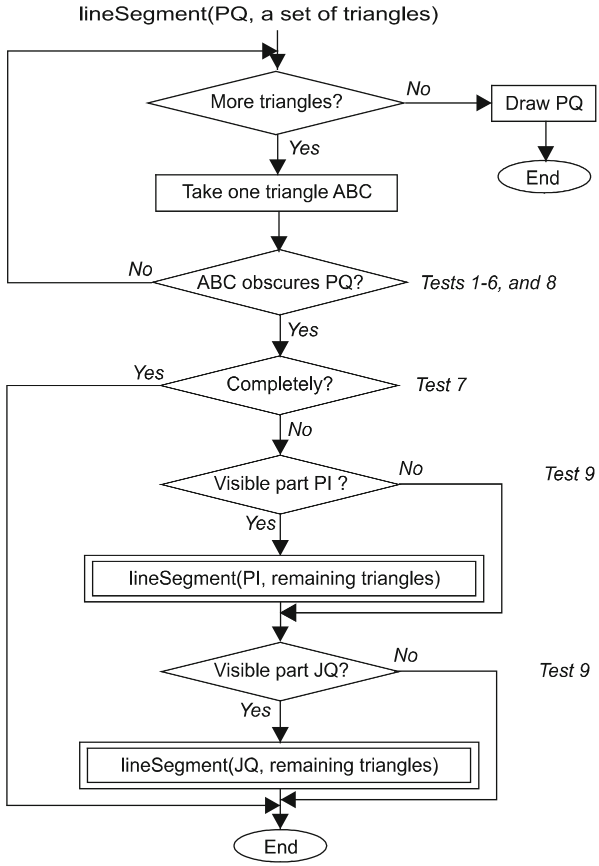


Fig. 6.3 Flowchart for *lineSegment* method

triangles. Note that in these cases the remaining triangles of the current loop are not applied to the whole line segment PQ anymore. The line PQ is drawn only if none of the triangles obscures it, that is, after the loop is completed.

The flowchart in Fig. 6.3 refers to tests 1–9. The first eight of them determine whether the given line segment PQ is completely or partially obscured by a triangle ABC. If ABC completely hides PQ, then, as far as PQ is concerned, we can ignore the remaining triangles. On the other hand, if ABC leaves PQ completely visible, we still have to test the visibility of PQ with regard to the remaining triangles. Only if none of all triangles hides PQ completely or partially can we conclude that PQ is really visible.

As soon as a triangle is found that partially hides PQ, test 9 will apply. One of the three cases (a), (b) and (c) shown in Fig. 6.4 must apply.

Fortunately, the intersection points I and J have been obtained in the previous case. If PI is visible, i.e. the cases in Fig. 6.4a, b, PI and the remaining triangles are sent to the recursive call (also shown in the flowchart of Fig. 6.3). Otherwise, if JQ is visible, i.e. the cases in Fig. 6.4b, c, JQ and the remaining triangles are sent to the

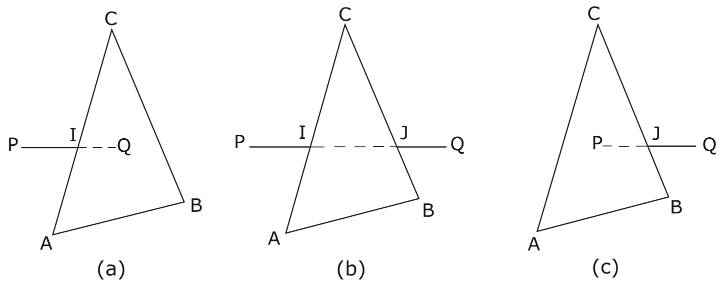


Fig. 6.4 Three cases where line segment PQ is partially blocked by triangle ABC

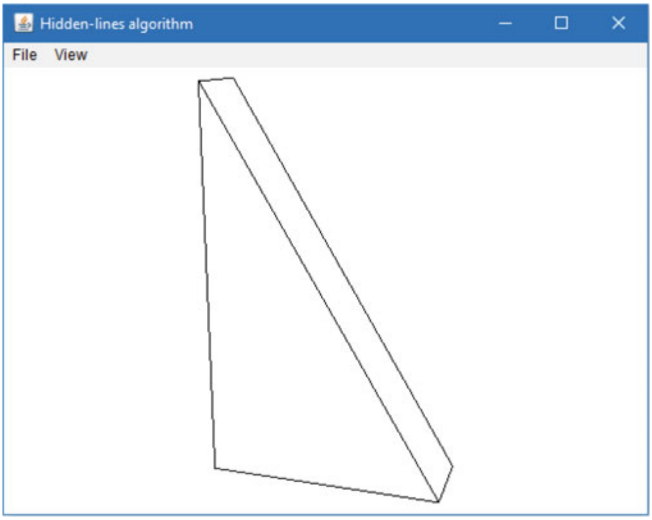


Fig. 6.5 Simple object displayed with *HLines.java*

recursive call. The details of the above nine tests and their Java implementations can be found in Appendix C. The program *HLines.java*, listed there, uses many classes that we have discussed in connection with program *Wireframe.java* of the previous chapter. Figure 6.5 shows the same object as that of Figs. 5.18, 5.19, and 5.20, but without three object edges that are hidden.

6.2 Backface Culling

Starting from this section, we will focus on the faces rather than on the edges of objects. For example, a cube is represented by its twelve edges in wireframe representation, but, for opaque objects, (in most cases) by three of its six faces. The three other faces, on the back and therefore known as *backfaces*, are invisible, so we can ignore them.

A backface can be detected by investigating the orientation of its vertices. To begin with, we specify each face counter-clockwise, when viewed from outside the object. For example, we denote the top face of the cube of Fig. 5.11 as the counter-clockwise vertex sequence 4, 5, 6, 7, or, for example, 6, 7, 4, 5, but not 6, 5, 4, 7, for that would be clockwise. In contrast, we can specify the bottom face of the cube as the sequence 0, 3, 2, 1, which is counter-clockwise when this face is viewed from the outside but clockwise in our perspective image. Here we see that the orientation of the bottom face in the image is different from that in 3D space, when the object is viewed from the outside (that is, from below). This is because this bottom face is a backface. We use this principle to tell backfaces from visible faces. The method *area*2 in the class *Tools*2*D*, discussed in Sect. 2.3, will now be very useful. Recall that the complete version of class *Tools*2*D* can be found in Sect. 2.3.

Note that we should use screen coordinates in the test we have just been discussing. If we used the x_e- and y_e-coordinates instead (ignoring the z_e-coordinate; see Fig. 5.4), this test about the orientation of the vertices would be equivalent to testing whether the normal vector, perpendicular to the face in question and pointing outward, would point more or less towards us or away from us, that is, whether that vector would have a positive or a negative z_e-component. This test would be correct with orthographic (or parallel) projection but not necessarily with central projection, which we are using. Exercise 6.5 deals with this subject in greater detail.

This time we will use the *Graphics* methods *setColor* and *fillPolygon*, to display each face, if it is visible, in a color that is unique for that face. Just before we do this (in the *paint* method) we test whether the face we are dealing with is visible; if it is not, it is a backface so that we can ignore it. To demonstrate that this really works, we will modify the viewpoint (by altering the angles θ and φ) each time the user presses a mouse button:

```
// Backface.java: A cube in perspective with backface culling.
// Uses: Point2D (Section 1.4), Point3D (Section 3.9),
//       Tools2D (Section 2.3).

import java.awt.*;
import java.awt.event.*;

public class Backface extends Frame {
   public static void main(String[] args) {new Backface();}

   Backface() {
      super("Press mouse button ...");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
```

```
      add("Center", new CvBackface());
      Dimension dim = getToolkit().getScreenSize();
      setSize(dim.width / 2, dim.height / 2);
      setLocation(dim.width / 4, dim.height / 4);
      setVisible(true);
   }
}

class CvBackface extends Canvas {
   int centerX, centerY;
   ObjFaces obj = new ObjFaces();
   Color[] color = {Color.blue, Color.green, Color.cyan,
                    Color.magenta, Color.red, Color.yellow};
   float dPhi = 0.1F;

   CvBackface() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            obj.theta += 0.1F;
            obj.phi += dPhi;
            if (obj.phi > 2 || obj.phi < 0.3)
               dPhi = -dPhi;
            repaint();
         }
      });
   }

   int iX(float x) {return Math.round(centerX + x);}
   int iY(float y) {return Math.round(centerY - y);}

   public void paint(Graphics g) {
      Dimension dim = getSize();
      int maxX = dim.width - 1, maxY = dim.height - 1,
            minMaxXY = Math.min(maxX, maxY);
      centerX = maxX / 2; centerY = maxY / 2;
      obj.d = obj.rho * minMaxXY / obj.objSize;
      obj.eyeAndScreen();
      Point2D[] p = new Point2D[4];
      for (int j = 0; j < 6; j++) {
         Polygon pol = new Polygon();
         Square sq = obj.f[j];
         for (int i = 0; i < 4; i++) {
            int vertexNr = sq.nr[i];
            p[i] = obj.vScr[vertexNr];
            pol.addPoint(iX(p[i].x), iY(p[i].y));
         }
```

```
         g.setColor(color[j]);
         if (Tools2D.area2(p[0], p[1], p[2]) > 0)
            g.fillPolygon(pol);
      }
   }
}

class ObjFaces { // Contains 3D object data of cube faces
   float rho, theta = 0.3F, phi = 1.3F, d;
   Point3D[] w;    // World coordinates
   Point3D[] e;    // Eye coordinates
                   // (e = wV where V is a 4 x 4 matrix)
   Point2D[] vScr; // Screen coordinates
   Square[] f;     // The six (square) faces of a cube.
   float v11, v12, v13, v21, v22, v23, v32, v33, v43,
         // Elements of viewing matrix V.
         xe, ye, ze, objSize;

   ObjFaces() {
      w = new Point3D[8];
      e = new Point3D[8];
      vScr = new Point2D[8];
      f = new Square[6];
      // Bottom surface:
      w[0] = new Point3D(1, -1, -1);
      w[1] = new Point3D(1, 1, -1);
      w[2] = new Point3D(-1, 1, -1);
      w[3] = new Point3D(-1, -1, -1);
      // Top surface:
      w[4] = new Point3D(1, -1, 1);
      w[5] = new Point3D(1, 1, 1);
      w[6] = new Point3D(-1, 1, 1);
      w[7] = new Point3D(-1, -1, 1);
      f[0] = new Square(0, 1, 5, 4); // Front
      f[1] = new Square(1, 2, 6, 5); // Right
      f[2] = new Square(2, 3, 7, 6); // Back
      f[3] = new Square(3, 0, 4, 7); // Left
      f[4] = new Square(4, 5, 6, 7); // Top
      f[5] = new Square(0, 3, 2, 1); // Bottom
      objSize = (float) Math.sqrt(12F);
         // distance between two opposite vertices.
      rho = 3 * objSize; // For reasonable perspective effect
   }
```

```
   void initPersp() {
      float costh = (float) Math.cos(theta),
            sinth = (float) Math.sin(theta),
            cosph = (float) Math.cos(phi),
            sinph = (float) Math.sin(phi);
      v11 = -sinth; v12 = -cosph * costh; v13 = sinph * costh;
      v21 = costh;  v22 = -cosph * sinth; v23 = sinph * sinth;
                    v32 = sinph;          v33 = cosph;
                                          v43 = -rho;
   }

   void eyeAndScreen() {
      initPersp();
      for (int i = 0; i < 8; i++) {
         Point3D p = w[i];
         float x = v11 * p.x + v21 * p.y;
         float y = v12 * p.x + v22 * p.y + v32 * p.z;
         float z = v13 * p.x + v23 * p.y + v33 * p.z + v43;
         Point3D Pe = e[i] = new Point3D(x, y, z);
         vScr[i] = new Point2D(-d * Pe.x / Pe.z, -d * Pe.y / Pe.z);
      }
   }
}

class Square {
   int nr[];
   Square(int iA, int iB, int iC, int iD) {
      nr = new int[4];
      nr[0] = iA; nr[1] = iB; nr[2] = iC; nr[3] = iD;
   }
}
```

If you do not use the Java *classpath* variable, you should make sure that the files *Point2D.class*, *Point3D.class* and *Tools2D.class* (or the corresponding *.java* files) are in the same directory as this program. We now initially obtain the image of a cube with red, green and blue visible faces, as shown in Fig. 6.6. By pressing the mouse button, the viewpoint changes, which has the same effect as a rotation of the cube. If we do this several times, the backfaces become visible one by one: yellow, cyan and magenta for the faces that are initially at the bottom, at the back and at the left, respectively.

We will use backface culling in the programs that follow. This technique is worthwhile because it reduces the number of polygons that may be visible drastically and it is inexpensive compared with some more time-consuming algorithms to be discussed later in this chapter. However, using only backface culling is not sufficient, since it does not work for non-convex solids, that is, any object that has at

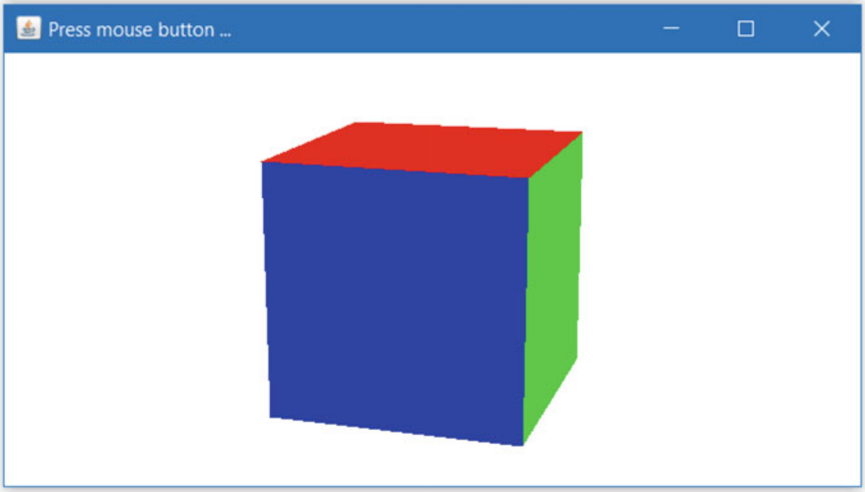


Fig. 6.6 Result of backface culling

least one non-convex polygon as its face. As Fig. 6.2 illustrates, the solid letter *L* is not convex, since the line that connects, for example, vertices 1 and 11 does not lie completely inside the object. In that example, rectangle 5-6-7-8 is not a backface, but it is only partly visible. It is also possible for a frontface (with vertex numbers in counter-clockwise order) to be completely invisible, since it may be hidden by another frontface that is nearer to the viewer. Backface culling may also fail in such cases.

6.3 Painter's Algorithm

It is an attractive idea to solve the hidden-face problem by displaying all polygons in a specific order, namely first the most distant one, then the second furthest, and so on, finishing with the one that is closest to the viewpoint. Painters sometimes follow the same principle, particularly when working with oil paintings. They typically start with the background and paint a new layer for objects on the foreground later, so that the overlapped parts of objects in the background, painted previously, are covered by the current layer and thus become invisible. This algorithm is therefore known as the *painter's algorithm*. It is based on the assumption that each triangle can be assigned a z_e-coordinate, which we can use to sort all triangles. One way of obtaining such a z_e-coordinate for a triangle ABC is by computing the average of the three z_e-coordinates of A, B and C. However, there is a problem, illustrated in Fig. 6.7. Each of the three triangles is partly obscured by another, so we cannot satisfactorily place them in any order, that is, from back to front (see also Figs. 6.12 and 6.13 for

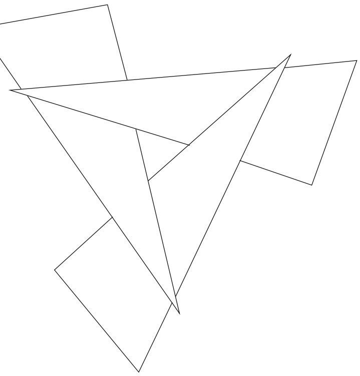


Fig. 6.7 Triangles, each of which partly obscures another

the similar case of three rectangular beams). Consequently, the naïve approach just suggested will fail in this case. Surprisingly, it gives good results in many other cases, such as the solid letter *L* in Fig. 6.2 and sphere and cone in Fig. 6.9. It is also very fast.

The program *Painter.java* is listed below. If you compile and run this program, all classes defined in the program files *Point2D.java, Point3D.java, Obj3D.java*, etc., must be available, as the comment below indicates. Recall our discussion of this subject at the end of Chap. 1.

```
// Painter.java: Perspective drawing using an input file that lists
//    vertices and faces. Based on the Painter's algorithm.
// Uses: Fr3D (Section 5.6) and CvPainter (Section 6.3),
//       Point2D (Section 1.4), Point3D (Section 3.9),
//       Obj3D, Polygon3D, Tria, Fr3D, Canvas3D (Section 5.6).
import java.awt.*;

public class Painter extends Frame {
   public static void main(String[] args) {
      new Fr3D(args.length > 0 ? args[0] : null, new CvPainter(),
            "Painter");
   }
}
```

In the above *Fr3D* constructor call, the first argument is a conditional expression to check if the user has used the option of specifying an input file as a program argument in the command line. (This is not required, since the user can also use the *File* menu to open input files). The second argument of the constructor call just mentioned creates an object of class *CvPainter*, which is listed below in the separate file *CvPainter.java*. The third argument specifies the window title *Painter* that will appear.

```
// CvPainter.java: Used in the file Painter.java.
import java.awt.*;
import java.util.*;

class CvPainter extends Canvas3D {
   private int maxX, maxY, centerX, centerY;
   private Obj3D obj;
   private Point2D imgCenter;

   Obj3D getObj() {return obj;}
   void setObj(Obj3D obj) {this.obj = obj;}
   int iX(float x) {return Math.round(centerX + x - imgCenter.x);}
   int iY(float y) {return Math.round(centerY - y + imgCenter.y);}

   void sort(Tria[] tr, int[] colorCode, float[] zTr, int l, int r) {
     int i = l, j = r, wInt;
     float x = zTr[(i + j) / 2], w;
     Tria wTria;
     do {
       while (zTr[i] < x) i++;
       while (zTr[j] > x) j--;
       if (i < j) {
         w = zTr[i]; zTr[i] = zTr[j]; zTr[j] = w;
         wTria = tr[i]; tr[i] = tr[j]; tr[j] = wTria;
         wInt = colorCode[i]; colorCode[i] = colorCode[j];
         colorCode[j] = wInt;
         i++;
         j--;
       }
       else if (i == j) {i++; j--;}
     } while (i <= j);
     if (l < j) sort(tr, colorCode, zTr, l, j);
     if (i < r) sort(tr, colorCode, zTr, i, r);
   }

   public void paint(Graphics g) {
      if (obj == null) return;
      obj.setSpecular(specularDesired);
         // specularDesired defined in Canvas3D
      Vector<Polygon3D> polyList = obj.getPolyList();
      if (polyList == null) return;
      int nFaces = polyList.size();
      if (nFaces == 0) return;
```

```
Dimension dim = getSize();
maxX = dim.width - 1; maxY = dim.height - 1;
centerX = maxX / 2; centerY = maxY / 2;
// ze-axis towards eye, so ze-coordinates of
// object points are all negative.
// obj is a java object that contains all data:
// - Vector w (world coordinates)
// - Array e (eye coordinates)
// - Array vScr (screen coordinates)
// - Vector polyList (Polygon3D objects)

// Every Polygon3D value contains:
// - Array 'nrs' for vertex numbers
// - Values a, b, c, h for the plane ax+by+cz=h.
// - Array t (with nrs.length-2 elements of type Tria)

// Every Tria value consists of the three vertex
// numbers iA, iB and iC.
obj.eyeAndScreen(dim);
// Computation of eye and screen coordinates.

imgCenter = obj.getImgCenter();
obj.planeCoeff(); // Compute a, b, c and h.

// Construct an array of triangles in
// each polygon and count the total number
// of triangles:
int nTria = 0;
for (int j = 0; j < nFaces; j++) {
   Polygon3D pol = polyList.elementAt(j);
   if (pol.getNrs().length < 3 || pol.getH() >= 0) continue;
 //  if (pol.triangulate(obj) != null)
 //     nTria += pol.getT().length;
   nTria += pol.triangulate(obj).length;
}
Tria[] tr = new Tria[nTria];
int[] colorCode = new int[nTria];
float[] zTr = new float[nTria];
int iTria = 0;
Point3D[] e = obj.getE();
Point2D[] vScr = obj.getVScr();

for (int j = 0; j < nFaces; j++) {
   Polygon3D pol = polyList.elementAt(j);
   if (pol.getNrs().length < 3 || pol.getH() >= 0) continue;
```

```
         int cCode =
         obj.colorCodePhong(pol.getA(), pol.getB(), pol.getC());
            g.setColor(new Color(cCode));

         Tria[] t = pol.getT();
         for (int i = 0; i < t.length; i++) {
            Tria tri = t[i];
            tr[iTria] = tri;
            colorCode[iTria] = cCode;
            float zA = e[tri.iA].z,
                  zB = e[tri.iB].z,
                  zC = e[tri.iC].z;
            zTr[iTria++] = zA + zB + zC;
         }
      }

      if (nTria > 0)
         sort(tr, colorCode, zTr, 0, nTria - 1);

      for (iTria = 0; iTria < nTria; iTria++) {
         Tria tri = tr[iTria];
         Point2D a = vScr[tri.iA],
                 b = vScr[tri.iB],
                 c = vScr[tri.iC];
         int cCodeAll = colorCode[iTria];
         g.setColor(new Color(cCodeAll));
         int[] x = {iX(a.x), iX(b.x), iX(c.x)};
         int[] y = {iY(a.y), iY(b.y), iY(c.y)};
         g.fillPolygon(x, y, 3);
      }
   }
}
```

We use a special method, *sort*, to sort the triangles; it is based on the well-known and efficient *quicksort* algorithm, discussed in detail in *Algorithms and Data Structures in C++* (Ammeraal 1996). Before calling *sort*, we build three arrays:

Array element	Type	Contains
tr[*iTria*]	*Tria*	The three vertex numbers of the triangle
colorCode[*iTria*]	*int*	Value between 0 and 255
zTr[*iTria*]	*float*	Value representing z_e-coordinate of triangle

For a given subscript value *iTria*, the three array elements in the first column of this table belong to the same triangle. We may regard them as members of the same record, of which *zTr* is the key. During sorting, whenever two elements *zTr*[*i*] and *zTr*[*j*] are swapped, we swap *tr*[*i*] and *tr*[*j*] as well as *colorCode*[*i*] and *colorCode*[*j*] at the same time. It seems reasonable to use the average of the three z_e-coordinates a triangle's vertices as the z_e-coordinate of that triangle, but we may simply use the sum instead of the average. Recall that the positive z_e-axis points towards us, so that all z_e values that we use are negative: the stronger negative it is, the further the triangle is. It follows that we have to paint the triangles in increasing order of their z_e-coordinates, or, equivalently, in decreasing order of their absolute values.

For each polygon, a color shade is computed by the method *colorCodePhong*, to be discussed in detail in Sect. 7.7. As the class *CvPainter* in the program *Painter.java* shows, there is a call to *colorCodePhong* in the method *paint* for every polygon just before entering the (inner) loop that, for a given polygon, deals with all its triangles. Within the loop, all the triangles of that polygon, their color code, and each triangle's distance, are stored in the three arrays *tr*, *colorCode* and *zTr*, discussed above. After this has been done for all polygons, the triangles are sorted on their distances stored in *zTr*, and then displayed in order of decreasing distance.

Next, we demonstrate the application of the painter's algorithm to some simple 3D objects, with the implementation of user operations. As Fig. 6.8 shows, we will allow the user to change the viewpoint E, characterized by its spherical coordinates. Immediately after an input file has been opened, these coordinates have the following (default) values:

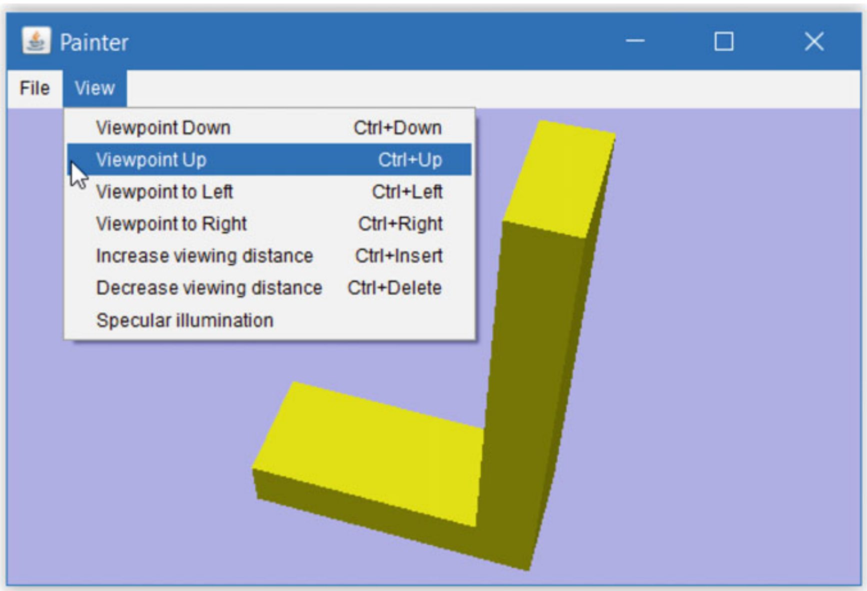


Fig. 6.8 Program *Painter.java* applied to file *letterL.dat*

$\rho = 3 \times$ the distance between two opposite vertices of a box in which the object fits

$\theta = 0.3$ radians ($\approx 17°$)

$\varphi = 1.3$ radians ($\approx 74°$)

If the user gives the menu command *Viewpoint Down* the value of φ is increased by 0.1 radians ($\approx 6°$). Similarly, *Viewpoint Up* decreases φ by 0.1 radians, while *Viewpoint Left* and *Viewpoint Right* decreases and increases θ by 0.1 radians, respectively. The same effects can be achieved by using one of the four arrow keys, ↓, ↑, ← and →, together with the Ctrl-key, as indicated in the menu. Figure 6.8 shows the screen when the input file *letterL.dat*, discussed in Sect. 6.1, has been read by program *Painter.java* and the user has used the *Viewpoint Up* command six times, so that we view the object from a higher viewpoint than in the initial situation. Recall that we saw the *View* menu also in Sect. 5.7 for the program *Wireframe*. It is also available in program *HLines* of Sect. 6.1. For the programs *Painter* and *ZBuf* (to be discussed in the next section) the *View* menu has the new menu item *Specular illumination*, which we will discuss in Chap. 7.

The *File* menu, not shown in Fig. 6.8, consists of the following commands:

```
Open   Ctrl+O
Exit   Ctrl+Q
```

If several input files are available, the user can switch to another object by using the *Open* command (or Ctrl + O), as usual. The simplicity of our file format makes it easy to generate such files by other programs, as discussed in Sect. 5.8 and Appendix D. In particular, mathematically well-defined solids, such as the sphere and the cone of Fig. 6.9, are very suitable for this. Incidentally, this example demonstrates that what we call a ‘three-dimensional object’ may consist of several solids. Note, however, that the input-file format requires that all vertices, in this example of both the sphere and the cone, must be defined in the first part of the file and all faces in the second. In other words, the line with the word *Faces* must occur only once in the file. This example also shows that curved surfaces can be approximated by a great many flat faces. In this example, there are altogether 4954 vertices and 5100 faces. Here the command *Viewpoint Down* has been used five times and the command *Viewpoint Right* twice.

An object will normally appear with a reasonable perspective effect, but the user will be able to increase or decrease the viewing distance by using menu commands or their shortcuts, Ctrl+Insert and Ctrl+Delete, as Fig. 6.8 shows.

As you can see in Figs. 6.8 and 6.9, the chosen background color is light blue like the sky, and all faces of the object are yellow. The source of light is far away, left at the top, so that the upper and the left faces are bright yellow, while those on the right and on the bottom are much darker.

To start the program, we can supply the (initial) input file as a program argument. For example, if the first 3D object to be displayed is given by the file *letterL.dat*, we can start the program by entering

```
java Painter letterL.dat
```

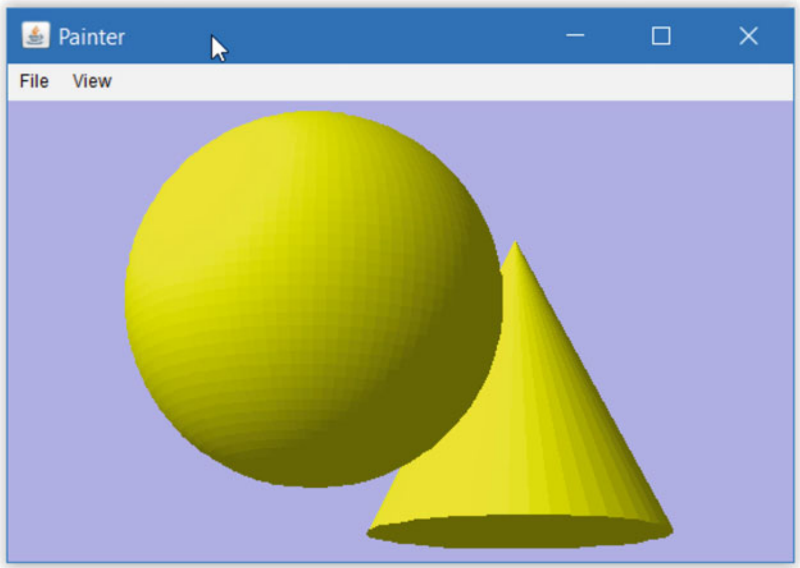


Fig. 6.9 Sphere and cone

Alternatively, we can enter

```
java Painter
```

and use the *Open* command in the *File* menu (or Ctrl+O) to specify the input file *letterL.dat*. With either way of starting the program, we can switch to another object by using this *Open* command.

As discussed at the beginning of this section, the painter's algorithm will fail in some cases, such as the three triangles in Fig. 6.7. We will therefore discuss another algorithm for hidden-face removal in the next section.

6.4 Z-Buffer Algorithm

If the images of two faces F_1 and F_2 overlap, we may consider two questions with regard to the correct representation of these faces:

1. Which pixels would be used for both F_1 and F_2 if each face were completely visible?
2. For each of these pixels, which of the corresponding points in F_1 and F_2 is nearer?

The Z-buffer algorithm deals with these questions in a general and elegant way. Recall that we are using an eye-coordinate system, with z-coordinates denoting the distance to the viewpoint E. We consider a points P in 3D-space and its corresponding projection P′ in 2D-space, where we use central projection with the viewpoint E as the center of projection. In other words, each line PE is a ray of light, intersecting the screen in P′. We are especially interested in such points that are the centers of pixels.

The Z-buffer algorithm is based on a large two-dimensional array that stores z-coordinates. Using the variable *dim* as the current canvas dimension, we define the following array, also known as a *Z-buffer*, for this purpose:

```
private float buf[][];
buf = new float[dim.width][dim.height];
```

We initialize array *buf* with values corresponding to points that are very far away. As before, we ignore backfaces. For each of the remaining faces we compute all pixels on the screen and their z-values on the face. For each pixel P′(ix, iy), we test whether the corresponding point P in 3D–space is nearer than *buf*[ix][iy] indicates. If it is, we put this pixel on the screen, using the color for the face in question, computed as in the previous section, while updating *buf*[ix][iy] at the same time:

```
For each face F (and its image, consisting of a set of pixels):
For each pixel P′(ix, iy), corresponding with a 3D point P of F:
  If P is nearer than the distance stored in buf[ix][iy] indicates, {
    set pixel P′ to the color for face F;
    update buf[ix][iy] so that it refers to the distance of P.
  }
```

In this discussion, the words *distance* and *near* refer to the z-coordinates in the eye-coordinate system. (Since we use no other 3D coordinates here, we simply write z instead of z_e here.) There are two aspects that make the implementation of the above algorithm a bit tricky:

1. Since the z-axis points towards us, the larger z is, the shorter the distance.
2. It is necessary to use $1/z$ instead of z for linear interpolation.

Let us take a look at this rather surprising point 2. Suppose we are given the z_e-coordinates of two points A and B in 3D-space and the central projections A′ and B′ of these points on the screen. Besides, some point P′ on the screen, lying on A′B′, is given and we have

$$x_P{}' = x_A{}' + \lambda(x_B{}' - x_A{}')$$
$$y_P{}' = y_A{}' + \lambda(y_B{}' - y_A{}')$$

where $x_P{}'$, and so on, are screen coordinates. We are then interested in the point P (in 3D-space) of which P′ is the central projection. Since we want to know how far P is away, our goal is to compute z_P. (After this, we can also compute the 3D coordinates $x_P = -x_P{}' z_P/d$ and $y_P = -y_P{}' z_P/d$, using Eqs. (5.7) and (5.8) of Sect. 5.3, where we wrote X and Y instead of x' and y'). Curiously enough, to compute this eye coordinate z_P by interpolation, we need to use the inverse values of the z-coordinates:

$$\frac{1}{z_P} = \frac{1}{z_A} + \lambda\left(\frac{1}{z_B} - \frac{1}{z_A}\right)$$

We will simply use this result here; it is discussed in more detail in Appendix A. Let us write

$$z_{Pi} = \frac{1}{z_P}$$

which we write as *zPi* (equal to 1/*zP*) in the program. Using the same convention ($z_{Ai} = 1/zA$, etc.) for other variables and writing *xA*, *yA* etc. for screen coordinates, we compute the centroid $D(x_D, y_D)$ along with its inverse z-value z_{Di} for each triangle ABC as follows:

```
xD = (xA + xB + xC)/3;
yD = (yA + yB + yC)/3;
zDi = (zAi + zBi + zCi)/3;
```

This centroid will be the basis for computing z_i-values for other points of the triangle by linear interpolation. To do this, we are interested in how much z_{Pi} increases if P moves one pixel to the right. This quantity, which we may write as $\partial z_i/\partial x$ or simply as *dzdx* in the program, is constant for the whole triangle. We will use this value, as well as its counterpart $dzdy = \partial z_i/\partial y$ indicating how much z_{Pi} increases if P moves one pixel upward, that is, if the screen coordinate y_P is increased by 1. It is useful to think of the triples x, y and z_i (where x and y are screen coordinates and $z_i = 1/z$) as points in a plane of an imaginary 3D space. We can then denote this plane as

$$ax + by + cz_i = k \tag{6.1}$$

Writing this in the form $z_i = (k - ax - by)/c$ and applying partial differentiation to z_i, we obtain

$$\frac{\partial z_i}{\partial x} = -\frac{a}{c} \tag{6.2}$$

$$\frac{\partial z_i}{\partial y} = -\frac{b}{c} \tag{6.3}$$

To find a, b and c, remember that (a, b, c) in Eq. (6.1) is the normal vector of the plane of triangle ABC (in the imaginary space we are dealing with). We define the vectors

$$\mathbf{u} = \mathbf{AB} = (u_1, u_2, u_3)$$
$$\mathbf{v} = \mathbf{AC} = (v_1, v_2, v_3)$$

where

$$\begin{aligned} u_1 &= x_B - x_A & v_1 &= x_C - x_A \\ u_2 &= y_B - y_A & v_2 &= y_C - y_A \\ u_3 &= z_{Bi} - z_{Ai} & v_3 &= z_{Ci} - z_{Ai} \end{aligned}$$

Then the vector product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

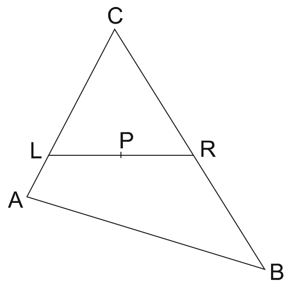
(see Sect. 2.2) is also perpendicular to triangle ABC, so that we can compute the desired values a, b and c of (6.1), (6.2) and (6.3) as the coefficients of **i**, **j** and **k**, respectively, finding

$$\begin{aligned} a &= u_2 v_3 - u_3 v_2 \\ b &= u_3 v_1 - u_1 v_3 \\ c &= u_1 v_2 - u_2 v_1 \end{aligned}$$

So much for the computation of $dzdx = \partial z_i / \partial x$ and $dzdy = \partial z_i / \partial y$, which we will use to compute z_{Pi} for each point P of triangle ABC, as we will see below.

To prepare for traversing all relevant *scan lines* for triangle ABC, such as LR in Fig. 6.10, we compute the y-coordinates *yTop* and *yBottom* of the vertices at the top and the bottom of this triangle. For all pixels that comprise triangle ABC on the screen, we have to compute the z-coordinate of the corresponding point in 3D space. In view of the enormous number of those pixels, we should do this as efficiently as possible. Working from the bottom of the triangle to the top, when dealing with a scan line, we traverse all its pixels from left to right. Figure 6.10

Fig. 6.10 Triangle ABC and scan line LR



shows a situation in which all pixels of triangle ABC below the scan line LR have already been handled.

For each scan line at level y ($yBottom \leq y \leq yTop$), we find the points of intersection L and R with triangle ABC as follows. We introduce the points I, J and K, which are associated with the triangle edges BC, CA and AB, respectively. Initially, we set the program variables xI, xJ and xK to 10^{30}, and $xI1$, $xJ1$ and $xK1$ to -10^{30}. Then, if y lies between yB and yC or is equal to one of these values, we compute the point of intersection of the scan line with BC, and we assign the x-coordinate of this point to both xI and $xI1$. In the same way we possibly update xJ, $xJ1$ for CA and xK and $xK1$ for AB. After this, each of the variables xI, xJ and xK is equal either to its original value 10^{30} or to the x-coordinate of the scan line in question with BC, CA and AB, respectively. The same applies to the other three variables, except that these may still have a different original value, -10^{30}. We can now easily find xL and xR:

$$xL = \min(xI, xJ, xK)$$
$$xR = \max(xI1, xJ1, xK1)$$

So far, we have been using floating-point, logical coordinates y, xL and xR. Since we have to deal with pixels, we convert these to the (integer) device coordinates iY, iXL and iXR as follows:

```
int iy = iY(y), iXL = iX(xL+0.5), iXR = iX(xR-0.5);
```

Adding 0.5 to xL and subtracting it from xR is done to prevent clashes between neighboring triangles of different colors: the pixel (iXR, y) belonging to triangle ABC should preferably not also occur as a pixel (iXL, y) of the right-hand neighbor of this triangle, because it would then not be clear which color to use for this pixel. Before entering the loop for all pixels iXL, $iXL + 1, \ldots, iXR$ for the scan line on level y, we compute the inverse z-value $zi = zLi$ for the pixel (iXL, y). Theoretically, this value is

$$z_{\mathrm{Li}} = z_{\mathrm{Di}} + (y - y_{\mathrm{D}})\frac{\partial z}{\partial y} + (x_{\mathrm{L}} - x_{\mathrm{D}})\frac{\partial z}{\partial x}$$

In the program we modify this a little, giving a little more weight to the centroid D of the triangle:

```
double zi = 1.01 * zDi + (y - yD) * dzdy + (xL - xD) * dzdx;
```

This modification is useful in some special cases of which we will give an example at the end of this section.

Starting at the left end (*iXL*, *y*) of a scan line with the above *z*-value $zi = z_{Li}$, we could now write the loop for this scan line as follows:

```
for (int x=iXL; x<=iXR; x++) {
   if (zi < buf[x][iy]) {    // '<' means 'nearer'
      g.drawLine(x, iy, x, iy);
      buf[x][iy] = (float)zi;
   }
   zi += dzdx;
}
```

Along a horizontal line LR, shown in Fig. 6.10, we compute the inverse *z*-coordinate, *zi*. If this is less than the contents of the array element *buf*[*x*][*y*], we put a pixel on the screen and update that array element.

The above test $zi < buf[x][y]$ may at first look confusing. Since the positive *z*-axis of the eye coordinate system points towards us, we have:

- the *larger* the *z*-coordinate of a point, the nearer this point is to the eye.

However, we are using inverse values $zi = 1/z$, so that the above is equivalent to

- the *less* the *zi*-value of a point, the nearer it is to the eye.

A complicating factor is that we are using negative *z*-coordinates, but the above also applies to negative numbers. The following example for two points P and Q will make the situation clear:

P nearby		Q far away
$z_P = -10$	$>$	$z_Q = -20$
$zi_P = 1/z_P = -0.1$	$<$	$zi_Q = 1/z_Q = -0.05$

Another curious aspect of the above fragment is that putting a pixel on the screen is done here by drawing a line of only one pixel. It is strange that Java does not supply a more elementary routine, say, *putPixel*, for this purpose. However, we can do much better by delaying this '*putPixel*' operation until we know for how many adjacent pixels it is to be used; in other words, we build horizontal line segments in memory, storing their leftmost *x*-values and displaying these segments if we can no longer extend it on the right. This implies that even if there were a *putPixel* method, we would not use it, but rather draw horizontal line segments, consisting of some pixels we have recently been dealing with. Instead of the above for-loop we will actually use an 'optimized' but functionally equivalent fragment, as we will discuss in a moment. The program *ZBuf.java* is listed below.

```
// ZBuf.java: Perspective drawing using an input file that
//    lists vertices and faces.
//    Z-buffer algorithm used for hidden-face elimination.
```

```
// Uses: CvZBuf (see below),
//       Point2D (Section 1.4), Point3D (Section 3.9) and
//       Obj3D, Polygon3D, Tria, Fr3D, Canvas3D (Section 5.6).
import java.awt.*;

public class ZBuf extends Frame {
   public static void main(String[] args) {
      new Fr3D(args.length > 0 ? args[0] : null, new CvZBuf(),
         "ZBuf");
   }
}
```

The class *CvZBuf* is defined in the following separate file:

```
// CvZBuf.java: Canvas class for ZBuf.java.
import java.awt.*;
import java.util.*;

class CvZBuf extends Canvas3D {
   private int maxX, maxY, centerX, centerY,
      maxX0 = -1, maxY0 = -1;
   private float buf[][];
   private Obj3D obj;
   private Point2D imgCenter;

   int iX(float x) {
      return Math.round(centerX + x - imgCenter.x);
   }

   int iY(float y) {
      return Math.round(centerY - y + imgCenter.y);
   }

   Obj3D getObj() {
      return obj;
   }

   void setObj(Obj3D obj) {
      this.obj = obj;
   }

   public void paint(Graphics g) {
      if (obj == null)
         return;
      obj.setSpecular(specularDesired);
```

```
      // specularDesired defined in Canvas3D
   Vector<Polygon3D> polyList = obj.getPolyList();
   if (polyList == null)
      return;
   int nFaces = polyList.size();
   if (nFaces == 0)
      return;
   float xe, ye, ze;

   Dimension dim = getSize();
   maxX = dim.width - 1;
   maxY = dim.height - 1;
   centerX = maxX / 2;
   centerY = maxY / 2;
   // ze-axis towards eye, so ze-coordinates of
   // object points are all negative. Since screen
   // coordinates x and y are used to interpolate for
   // the z-direction, we have to deal with 1/z instead
   // of z. With negative z, a small value of 1/z means
   // a small value of |z| for a nearby point. We there-
   // fore begin with large buffer values 1e30:
   if (maxX != maxX0 || maxY != maxY0) {
      buf = new float[dim.width][dim.height];
      maxX0 = maxX;
      maxY0 = maxY;
   }
   for (int iy = 0; iy < dim.height; iy++)
      for (int ix = 0; ix < dim.width; ix++)
         buf[ix][iy] = 1e30F;

   obj.eyeAndScreen(dim);
   imgCenter = obj.getImgCenter();
   obj.planeCoeff(); // Compute a, b, c and h.
   Point3D[] e = obj.getE();
   Point2D[] vScr = obj.getVScr();

   for (int j = 0; j < nFaces; j++) {
      Polygon3D pol = polyList.elementAt(j);
      if (pol.getNrs().length < 3 || pol.getH() >= 0)
         continue;
      int cCode = obj.colorCodePhong(
         pol.getA(), pol.getB(), pol.getC());
      g.setColor(new Color(cCode));

      pol.triangulate(obj);
```

```
      Tria[] t = pol.getT();
      for (int i = 0; i < t.length; i++) {
         Tria tri = t[i];
         int iA = tri.iA, iB = tri.iB, iC = tri.iC;
         Point2D a = vScr[iA], b = vScr[iB], c = vScr[iC];
         double zAi = 1 / e[iA].z, zBi = 1 / e[iB].z,
               zCi = 1 / e[iC].z;
         // We now compute the coefficients a, b and c
         // (written here as aa, bb and cc)
         // of the imaginary plane ax + by + czi = h,
         // where zi is 1/z (and x, y and z are
         // eye coordinates. Then we compute
         // the partial derivatives dzdx and dzdy:
         double u1 = b.x - a.x, v1 = c.x - a.x,
                u2 = b.y - a.y, v2 = c.y - a.y,
                cc = u1 * v2 - u2 * v1;
         if (cc <= 0)
            continue;
         double xA = a.x, yA = a.y, xB = b.x, yB = b.y,
            xC = c.x, yC = c.y,
            xD = (xA + xB + xC) / 3,
            yD = (yA + yB + yC) / 3,
            zDi = (zAi + zBi + zCi) / 3,
            u3 = zBi - zAi, v3 = zCi - zAi,
            aa = u2 * v3 - u3 * v2, bb = u3 * v1 - u1 * v3,
            dzdx = -aa / cc, dzdy = -bb / cc,
            yBottomR = Math.min(yA, Math.min(yB, yC)),
            yTopR = Math.max(yA, Math.max(yB, yC));
         int yBottom = (int) Math.ceil(yBottomR),
             yTop = (int) Math.floor(yTopR);

         for (int y = yBottom; y <= yTop; y++) {
            // Compute horizontal line segment (xL, xR)
            // for coordinate y:
            double xI, xJ, xK, xI1, xJ1, xK1, xL, xR;
            xI = xJ = xK = 1e30;
            xI1 = xJ1 = xK1 = -1e30;
            if ((y - yB) * (y - yC) <= 0 && yB != yC)
               xI = xI1 = xC + (y-yC)/(yB-yC) * (xB-xC);
            if ((y - yC) * (y - yA) <= 0 && yC != yA)
               xJ = xJ1 = xA + (y-yA)/(yC-yA) * (xC-xA);
```

```
                if ((y - yA) * (y - yB) <= 0 && yA != yB)
                   xK = xK1 = xB + (y-yB)/(yA-yB) * (xA-xB);
                // xL = xR = xI;
                xL = Math.min(xI, Math.min(xJ, xK));
                xR = Math.max(xI1, Math.max(xJ1, xK1));
                int iy = iY(y), iXL = iX((float) (xL + 0.5)),
                      iXR = iX((float) (xR - 0.5));
                double zi =
                   1.01* zDi + (y-yD)*dzdy + (xL-xD)*dzdx;
                /*
                   for (int x=iXL; x<=iXR; x++) {
                      if (zi < buf[x][iy]) { // < is nearer
                         g.drawLine(x, iy, x, iy);
                         buf[x][iy] = (float)zi;
                      }
                      zi += dzdx; }
                */
                // The above comment code is optimized below:
                // ---
                boolean leftmostValid = false;
                int xLeftmost = 0;
                for (int ix = iXL; ix <= iXR; ix++) {
                   if (zi < buf[ix][iy]) { // < means nearer
                      if (!leftmostValid) {
                         xLeftmost = ix;
                         leftmostValid = true;
                      }
                      buf[ix][iy] = (float) zi;
                   } else
                   if (leftmostValid) {
                      g.drawLine(xLeftmost, iy, ix - 1, iy);
                      leftmostValid = false;
                   }
                   zi += dzdx;
                }
                if (leftmostValid)
                   g.drawLine(xLeftmost, iy, iXR, iy);
                // ---
             }
          }
       }
    }
}
```

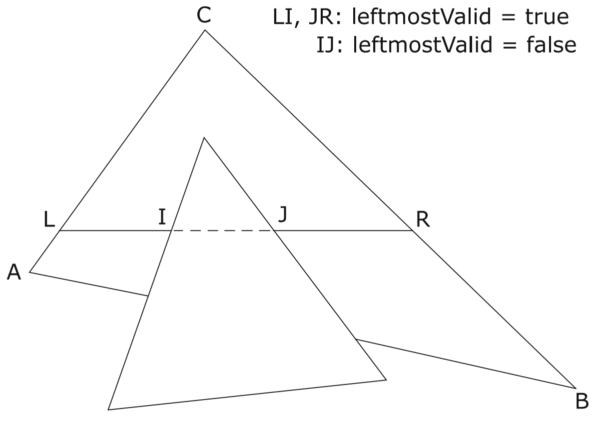


Fig. 6.11 Illustration of more efficient version

Almost at the end of this program, the fragment between the two lines // – is a more efficient version than the for-loop in the comment that precedes it. In that for-loop, each line LR is drawn pixel by pixel. In the more efficient version that follows, however, the line LR is drawn by one or more line segments depending on how much the line LR is blocked by the triangles that are nearer. In the best case, where no triangle is in front of the line, the line is drawn by a simple call to the *drawLine* method. Scanning through each horizontal line from left to right, we use a Boolean variable *leftmostValid* to record if we have found a leftmost point that is not blocked by any other triangle. If so, we keep scanning and updating the Z-buffer until we encounter the first point that is further away than the corresponding Z-buffer value, such as *I* in Fig. 6.11. We then draw the line up to the point just before that blocked point (*I*). We meanwhile set *leftmostValid* to false, entering the line segment, such as *IJ* in Fig. 6.11. Continuing scanning through until the first visible point, such as *J*, is found, we record *J* as the new leftmost point and set *leftmostValid* to true. This process continues to reach R, and the last line segment, such as *JR*, is drawn if it is not blocked.

Figure 6.12 demonstrates that the Z-buffer algorithm can also be used in cases in which the painter's algorithm fails. The latter is illustrated by Fig. 6.13. (See also the three triangles in Fig. 6.7).

In Appendix D (Sect. D.5) we will develop a program that enables us to display the surfaces of functions of two variables. Files generated by this program are also accepted by the program *ZBuf.java*, as Fig. 6.14 demonstrates:

Here we are using faces each with two sides, as discussed in Sect. 5.5. The way these are used here requires a correction in the computation of *zi*. Recall that we have introduced the factor 1.01 in the following statement:

```
double zi =  1.01 * zDi + (y - yD) * dzdy + (xL - xD) * dzdx;
```

Without this factor, some incorrect dark pixels would appear on the boundary, also referred as the *silhouette*, of the (yellow) object and the (light blue) background.

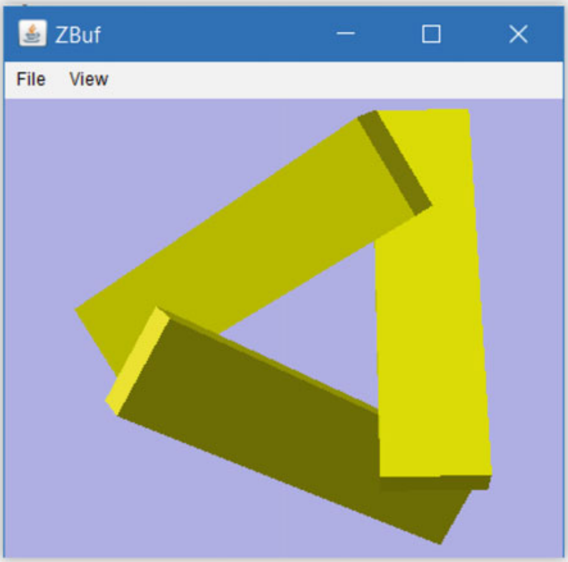


Fig. 6.12 Z-buffer algorithm applied to three beams

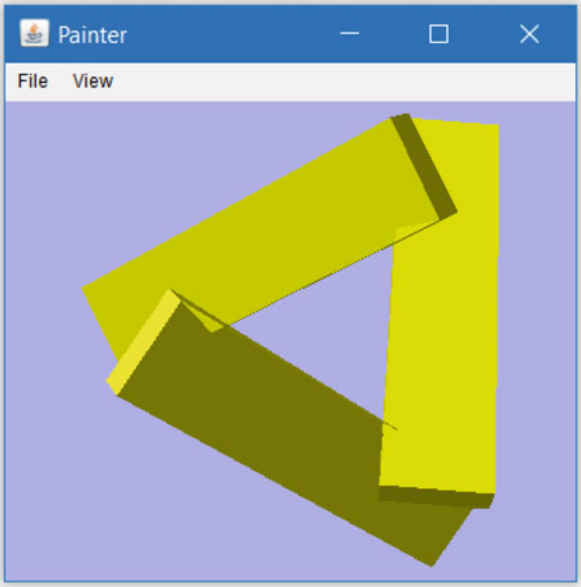


Fig. 6.13 Painter's algorithm failing

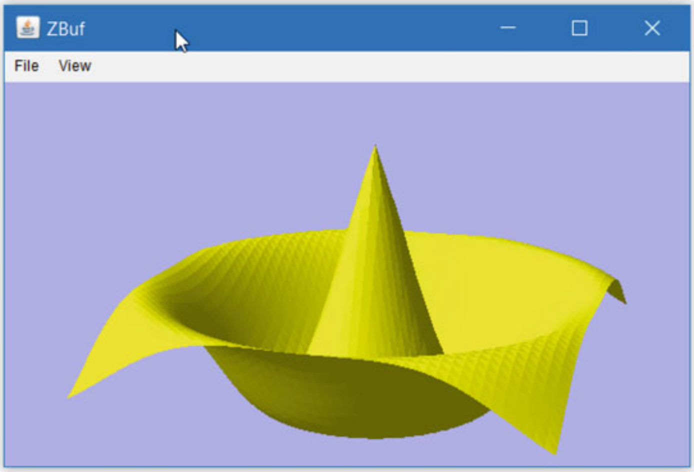
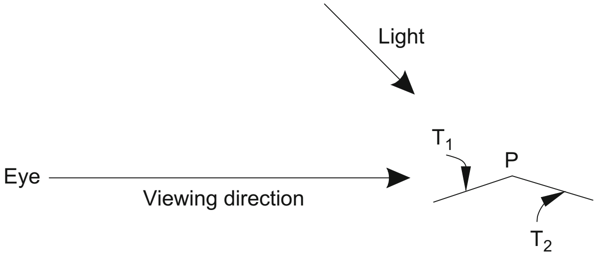


Fig. 6.14 Function of two variables

Fig. 6.15 Point P belonging to triangles of very different colors



To understand this, we note that in Fig. 6.15 point P lies on the boundary of two triangles T_1 and T_2.

The nearer triangle, T_1, is visible and its color is bright yellow because it lies on the upper side of the surface. Triangle T_2 lies on the lower side and would appear almost black on the screen if it were not obscured by T_1. Point P, belonging to both triangles, is used twice to determine the color of the corresponding pixel on the basis of the *zi* value of P. In both cases, this value is the same or almost the same, so that it is not clear which color will be used for this pixel. To solve this problem, we use the factor 1.01 instead of 1 in the above computation of *zi*. Because of this, the *zi*-value for P when taken as a point of T_1 will now be slightly less than when P is regarded as a point of T_2. As a result, the light yellow color will be used for the pixel in question.

Exercises

6.1. Use a normal editor or text processor to create a data file, to be used with program *HLines.java*, for the object consisting of both a horizontal square and a vertical line through its center, as shown twice in Fig. 6.16. On the left the viewpoint is above the square (as it is by default), while the situation on the right, with the viewpoint below the square, has been obtained by using the *Viewpoint Down* command of the *View* menu (or by pressing Ctrl + ↓).

6.2. Hidden-line removal works correctly for the problem in Exercise 6.1 because the vertical line passes through edges of the triangles produced by triangulation of the square. Change the horizontal square to a horizontal triangle and use *HLines.java* to display the object with the same vertical line through the triangle. The object may not be displayed properly like those in Fig. 6.16. Explain why and find a simple solution.

6.3. The same as Exercise 6.1 for Fig. 6.17, which shows two very thin square rings. Again, the object is shown twice to demonstrate that there are four potentially visible faces.

6.4. Write a program *HLinesDashed.java* similar to *HLines.java* but, instead of omitting hidden lines, draws them as dashed lines. For example, when applied to the file *letterA.dat*, discussed in Sect. 5.5, it produces the result of Fig. 6.18 (after changing the viewpoint). An easy way of doing this is by letting the method *lineSegment* in your class *CvHLinesDashed* first draw the whole line PQ as a dashed line, so these dashed lines, or parts of them, will later be overwritten by normally drawn lines if the line segments in question

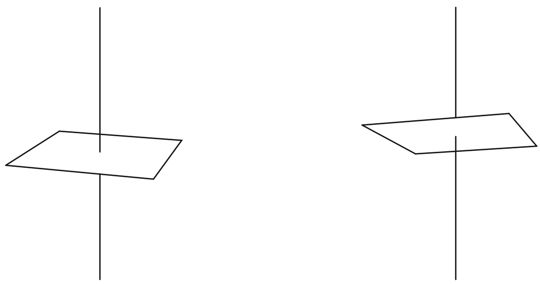


Fig. 6.16 Line passing through the center of a square

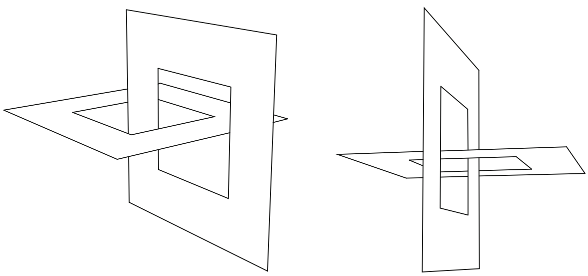


Fig. 6.17 Two thin square rings

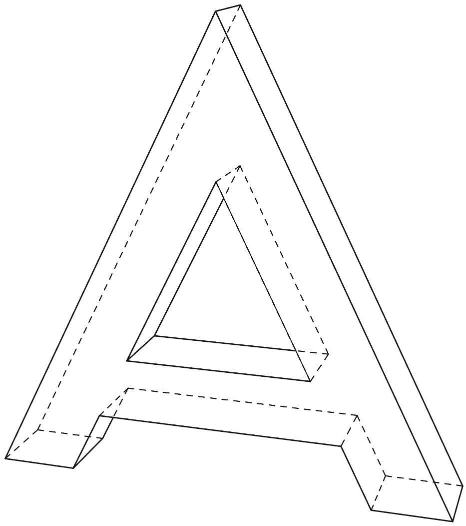


Fig. 6.18 Hidden lines represented by *dashed lines*

happen to be visible. Note that this should happen only for calls to *lineSegment* from the *paint* method, not for recursive calls. You can use a method *dashedLine* similar to that of Exercise 1.5.

6.5. The problem of backface culling cannot generally be solved by testing whether the z_e-component of the normal vector of a face in question is positive or negative. Recall that we are using the equation

$$ax + by + cz = h$$

for the plane of each face, in which $\mathbf{n} = (a, b, c)$ is the normal vector of that face, pointing outward. Using also the vector $\mathbf{x} = (x, y, z)$, we can write the above equation as

$$\mathbf{n} \cdot \mathbf{x} = h$$

Figure 6.19 shows the geometrical interpretation of $\mathbf{n}$, $\mathbf{x}$ and h. For a visible face, the inner product h is negative because $\mathbf{n}$ and $\mathbf{x}$ point in opposite directions. By contrast, h is positive for a backface.

Although in most cases c, the third component of the vector $\mathbf{n}$, is positive for a visible face and negative for a backface, there are situations in which this is not true. Give an example of such a situation, with a detailed explanation. How does backface culling based on the signs of h and (incorrectly) of c relate to that based on the orientation of point sequences?

6.6. Apply animation with double buffering to a cube, as in Exercise 5.4, but use colored faces.

6.7. Apply animation to two colored cubes (see Exercises 5.5 and 6.2. Even if one partially hide the other, this problem can be solved by backface culling,

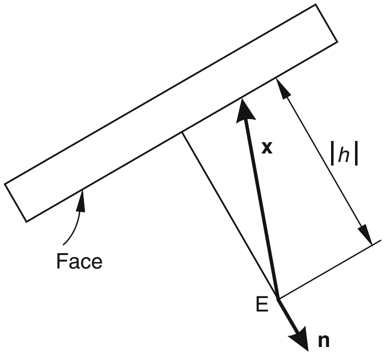


Fig. 6.19 Geometrical interpretation of **x**, **n** and h, where h is negative

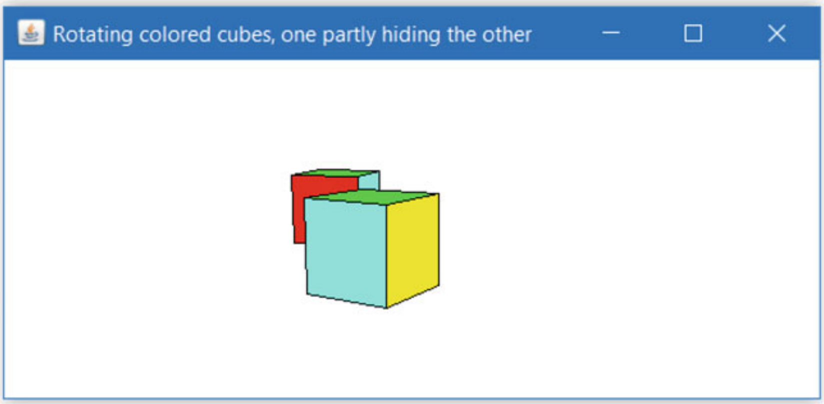


Fig. 6.20 Two rotating cubes, generated by backface culling

provided the farther cube is drawn before the nearer one. This is demonstrated in Fig. 6.20; here the Graphics methods *fillPolygon* and *drawPolygon* are applied to each visible face. By drawing the edges of such faces in black, we make sure that we can clearly distinguish the colored faces if, in reproductions, shades of gray are used instead of colors.

6.8. In the program file *Obj3D.java*, the direction of the light vector (*sunX*, *sunY*, *sunZ*) pointing to the source of light was arbitrarily chosen. Even without moving the object (by rotating it, for example), we can obtain an exciting effect by using animation to change this vector, so that the source of light rotates. Extend the program *Painter.java* to realize this. An easy way of doing this is by using spherical coordinates *sunTheta* and *sunPhi* and radius 1 for a light vector of unit length, similar to the spherical coordinates θ, φ and ρ, shown in Fig. 5.3, and increasing, say, *sunTheta* by 0.02 radians every 50 ms.

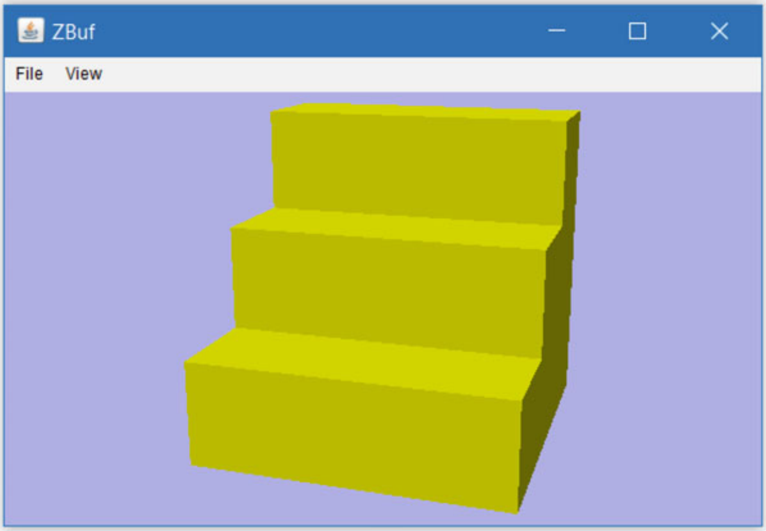


Fig. 6.21 Steps

6.9. Write a 3D data file (using a text editor) for the steps shown in Fig. 6.21. Apply the programs *Painter.java* and *ZBuf.java* to it. Can you write a program to generate such data files? In such a program, the number of steps and the dimensions should preferably be variable.

Chapter 7
Color, Texture, and Shading

Making computer graphics interesting and useful for a wide range of applications, color is a complex topic and may be interpreted in the context of visual psychology, physiology, or optics. In computer graphics, we will not discuss the physical properties and principles of color, rather, we focus on how to present colors in a way that they closely match those in the nature. We will introduce the RGB (Red, Green, Blue) model and its representation inside modern computers and the alternative but also commonly used HSL (Hue, Saturation, Luminance) model. The chapter then discusses the blending approach to model the transparency effects and how texture and lighting models are implemented.

7.1 Color Theories

There are mainly two competing color theories, i.e. trichromatic color theory and opponent color theory.

According to the trichromatic color theory, the human eye contains three types of light-sensitive receptors called *cones* and *rods*. One type of the receptor is most sensitive to red light, one type is most sensitive to green light, and one type is most sensitive to blue light. All the other colors could be obtained by mixing these three colors. Opponent color theory, on the other hand, states that human eyes could perceive light in three opposing components, i.e., light vs. dark, red vs. green, and blue vs. yellow. One could not sense the mixtures of red and green, or blue and yellow. Therefore, in reality, there is no such a color perceivable by human that is reddish green, or bluish yellow.

Computer graphics is based on the trichromatic color theory, adopting red, green, and blue (RGB) as the three primary colors. The trichromatic color theory is therefore often referred to as the RGB color system. Based on the RGB system, all perceivable colors could be represented in a three-dimensional space as an RGB cube, illustrated in Fig. 7.1. Every color could now be mapped onto a position in

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2_7

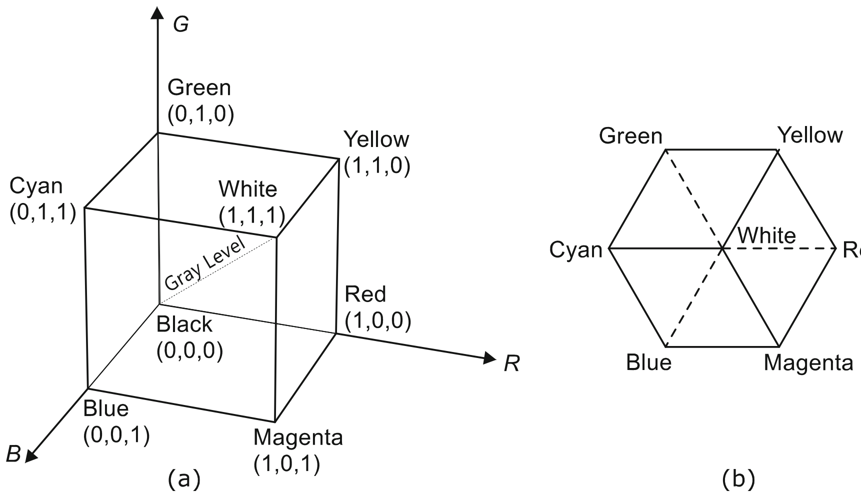


Fig. 7.1 (**a**) The 3D space for the RGB color model. (**b**) Viewed along principal diagonal

this 3D space, expressed as (r, g, b), where r, g, and b are real numbers between 0 and 1. For example, (0.3, 0.8, 0.5) is a point in the 3D space, uniquely representing a specific color.

Any given color C could then be expressed as

$$C = rR + gG + bB$$

Or simply

$$C = (r, g, b)$$

where r, g, and b represent the intensities of red, green, and blue respectively. This implies that every color is a linear combination of red, green, and blue.

The combination of two colors $C_1 = (r_1, g_1, b_1)$ and $C_2 = (r_2, g_2, b_2)$ can be written as

$$(r_1 + r_2)R + (g_1 + g_2)G + (b_1 + b_2)B$$

The RGB system matches the physical design of computer monitors, each made with a pattern of three types of phosphor emitting red, green, and blue light upon being excited by an electron beam. Each phosphor emits light intensities, expressed by r, g, and b as above, based on the energy of the electron beam that is directed at it. Note that the human perception of light and real-world colors are far more complex than what the RGB system could represent. The RGB system is so far the best and most feasible model for computer graphics. Some colors perceivable by the human eye could not be represented by the RGB system, nor be printed on paper.

7.2 Additive and Subtractive Colors

To explain the operation mechanism and relationships of colors, we consider the two types of color mixing: *additive* and *subtractive*. The behavior of light mixtures could be interpreted by *additive color*, and the behavior of paint, ink, dye, or pigment mixtures, could be interpreted by *subtractive color*. Additive and subtractive other systems have three primary colors which usually cause confusion for beginners in computer graphics. We will explain these two systems of colors in details in this section.

Additive Color

We start by considering the additive color system as the one without any light and thus having a black background. An additive color is the result of adding different amounts of red, green, and blue together. The mixture of the maximal intensities of all the three primary colors should generate white, as shown in Fig. 7.2a as the overlap of all the three circles. Equal mixing of two of the three primary colors would generate secondary colors, i.e. cyan, magenta, and yellow, which are shown as three overlapped areas, each between two primary colors.

The additive color mixing process is like adding together the colors in light to create new colors. This is exactly the principle in televisions, that mixes red light, green light, and blue light to generate an image. Apart from television, display monitors and projectors are all based on the additive color principle. Since the intensity information for each of the three colors is preserved, the image color is preserved as well.

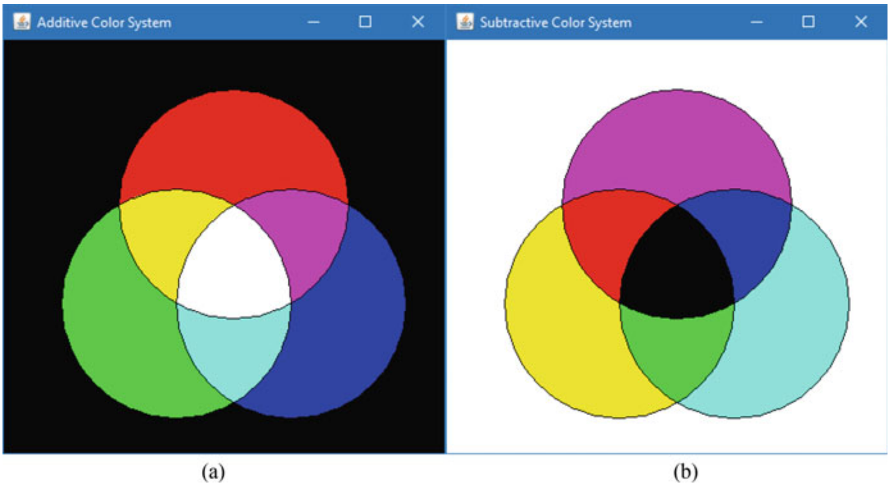


Fig. 7.2 Additive (**a**) and subtractive (**b**) color systems

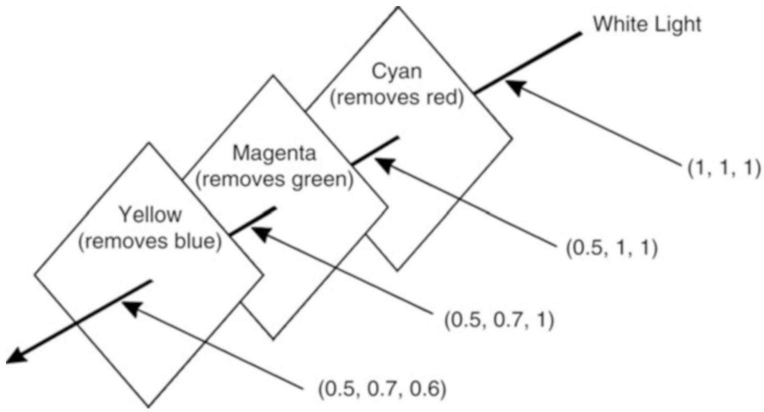


Fig. 7.3 The subtracting process in the subtractive color system

The three primary colors in light correspond to the red, green, and blue cones in the human eye, and thus are also the primary colors of the additive color system.

Subtractive Color

The subtractive color system is usually used in arts and design. It works just like mixing of color paints. The word "subtractive" here may be confusing as one may think of mixing color paints is like adding them up. The colors on objects we perceive are due to the corresponding colors in the light and we may consider to start with full light which is essentially white. All unneeded colors are subtracted (or filtered out as shown in Fig. 7.3) without entering our eyes.

Now we can consider the relationship between additive colors and subtractive colors. The three secondary colors in additive color system, i.e. cyan, magenta, and yellow (CMY), are obtained by subtracting the corresponding primary colors, and are therefore the primary colors in the subtractive color system.

Consider Fig. 7.3, cyan, magenta, and yellow (CMY) may be considered three primary colors in front of a white light source, after other colors have been filtered out from white. This principle can be explained by inserting three filters between a white light source (1, 1, 1) and a human eye. We assume the level of needed CMY mixing is expressed as (0.5, 0.3, 0.4). The first cyan filter removes 50% of red from white, yielding the RGB expression of (0.5, 1, 1) for cyan. The cyan light then passes through the magenta filter that removes 30% of green, yielding the RGB expression of (0.5, 0.7, 1) for magenta. Finally, magenta passes through the yellow filter that removes 40% of blue, yielding the RGB expression of (0.5, 0.7, 0.6) for yellow. The RGB expression of (0.5, 0.7, 0.4) is equivalent to the color of (0.5, 0.3, 0.4) in the CYM system, which is exactly what we needed.

Considering the cost and practicality, most printers support CMYK color, where K refers to the level of black in the color mix. In theory, equal mix of three primary colors would generate black, though in practice, no true black can be created. Using the pure black ink for printers is apparently cheaper than combining three primary colors, apart from providing true black. The amount of ink needed for black is also minimized.

Since the RGB system and CMY system complement each other, their relationships could be simply expressed as the following:

$$(c, m, y) = (1, 1, 1) - (r, g, b)$$

where (1, 1, 1) represents white in the RGB space; inversely,

$$(r, g, b) = (1, 1, 1) - (c, m, y)$$

where (1, 1, 1) represents black in the CMY space.

In summary, additive color system uses red, green, and blue (RGB) to represent the colors of real world objects. Subtractive color system uses cyan, magenta, and yellow (CMY) to represent the colors of light. The primary colors of the subtractive color system are the secondary colors of the additive color system. The additive RGB color system is adopted in computer graphics and displayed on computer monitors, directly supported by the hardware.

Application of Color Systems

The art community has studied various color theories, such as that of Munsell [17], long before the computer graphics field. Artists use color as one of their most essential means of conveying ideas. The Munsell [18] color system specifies colors in three dimensions, i.e. hue, value, and chroma, and suggests how to make balanced and harmonious color combinations. Some artists tend to choose colors for their psychological impact, for example, purple to convey horror, pink for joy, and green for envy. Kandinsky calls black the symbol of death and white that of birth. He also systematically associates colors with lines and basic geometric shapes. Media research provides other colors' emotional connotations, for example, lime green and avocado indicating nausea, yellow combined with black implying a warning and sometimes power, and blue representing the most likable color.

As a direct application of computer graphics, visualization uses color to encode various data attributes, according to human color perception. The following example demonstrates that information visualization can usefully apply our commonly understood color system used in painting, i.e. the subtractive CMY system. Art students, however, usually like to use the three primary colors they are familiar with, i.e. red, yellow, and blue (RYB). To assist the general public who are typically used to the color concepts in painting, Gossett and Chen [12] proposed using subtractive color system with red, yellow, and blue as primary colors for visualizing multi-attribute information. For example, to visualize the presence of three different racial populations (traits) in a state with one color representing each trait, the RGB map (in Figure 3 of Zhang [26]) displays Arizona as white due to the presence of all the three traits in the state. Common sense would then suggest that white implies the absence of any trait. In contrast, the RYB map displays a more meaningful brown color for Arizona. They use an RYB interpolation cube to convert from RGB colors to the corresponding RYB colors.

7.3 RGB Representation

In modern raster display systems, the number of different colors that can be supported depends on the display storage and screen resolution. The display storage is also called the *frame buffer*, where each frame is of the size of the entire screen. There are two approaches in storing colors in the frame buffer. The first approach stores the color codes of red, green, and blue directly into the frame buffer, while the second approach uses a color look up table (CLUT) to store color codes, which could be uniquely retrieved based on the positions of individual pixels on screen.

In the first approach, the size of the frame buffer is exactly the same as that of the screen, so that there is one-to-one correspondence between the pixels in the two. Therefore, retrieving a color code is extremely fast. Assuming each pixel has only one bit to encode colors, only two colors could be represented, such as black and white. The most basic color display monitor has to have at least three bits per pixel to represent colors, as listed in Table 7.1 for a simple color scheme. In the following, we will explain the RGB color display principle using this simple color coding scheme.

Each of the 3 bit positions is used to control the display intensity of the corresponding electron gun in an RGB display monitor. In this case, the intensity is either 1 or 0, i.e. on or off. The eight color codes represent exactly the eight vertices of the RGB cube in Fig. 7.1. Adding more bits per pixel to the frame buffer increases the number of colors that can be displayed on the screen. For example, with 6 bits per pixel, 2 bits can be used for each color. This would provide 4 different

Table 7.1 Given 3 bits per pixel dedicated to colors, one possible color coding scheme

Color code	Red	Green	Blue	Displayed color
0	0	0	0	Black
1	0	0	1	Blue
2	0	1	0	Green
3	0	1	1	Cyan
4	1	0	0	Red
5	1	0	1	Magenta
6	1	1	0	Yellow
7	1	1	1	White

intensities for each of the three primary colors, making a total of $64(=4 \times 4 \times 4)$ colors for each pixel. This also implies more storage required for the frame buffer.

The following program draws three circles of the primary colors (Red, Green and Blue) of the additive color system, together with their intersected regions displaying the secondary colors and white (when all the three bits are 1). The displayed colors, as shown in Fig. 7.2a, have the color codes of 1–7 as illustrated in Table 7.1. The program drawing color interactions in the subtractive color system could be similarly written and the corresponding colors are shown in Fig. 7.2b.

Although our monitors are capable of displaying many more colors, this particular program chooses to display only 7 most basic colors, without making the full use of the available bits.

```
// ColorCirles.java: draw three circles with primary colors Red,
// Green and Blue and their intersections with additive colors.
import java.awt.*;
import java.awt.event.*;
import java.awt.geom.*;

public class ColorCircles extends Frame {
   public static void main(String[] args){new ColorCircles();}
   ColorCircles() {
   super("Additive Color System");
   addWindowListener(new WindowAdapter(){
       public void windowClosing(WindowEvent e){System.exit(0);}});
       setSize(400, 400);
       add("Center", new CvColorCircles());
       setVisible(true);
   }
}

class CvColorCircles extends Canvas {
   public void paint(Graphics g) {
      super.paint(g);
      final int RED = 0xFF0000, GREEN = 0x00FF00, BLUE = 0x0000FF;
      Shape circleTop = new Ellipse2D.Double(100, 43, 200, 200);
      Shape circleLeft = new Ellipse2D.Double(50, 130, 200, 200);
      Shape circleRight = new Ellipse2D.Double(150, 130, 200, 200);
      Area top = new Area(circleTop);
      Area left = new Area(circleLeft);
      Area right = new Area(circleRight);
      Area intersectTopLeft = new Area(top);
      intersectTopLeft.intersect(left);
      Area intersectLeftRight = new Area(left);
      intersectLeftRight.intersect(right);
      Area intersectTopRight = new Area(top);
```

```
        intersectTopRight.intersect(right);
        Area intersectCenter = new Area(intersectTopLeft);
        intersectCenter.intersect(right);
        setBackground(Color.black);
        Graphics2D g2 = (Graphics2D)g;
        g2.setColor(new Color(RED));
        g2.fill(top);
        g2.setColor(new Color(GREEN));
        g2.fill(left);
        g2.setColor(new Color(BLUE));
        g2.fill(right);
        g2.setColor(new Color(RED + GREEN));        // Yellow
        g2.fill(intersectTopLeft);
        g2.setColor(new Color(GREEN + BLUE));       // Cyan
        g2.fill(intersectLeftRight);
        g2.setColor(new Color(RED + BLUE));         // Magenta
        g2.fill(intersectTopRight);
        g2.setColor(new Color(RED + GREEN + BLUE)); // White
        g2.fill(intersectCenter);
        g2.setColor(Color.black);
        g2.draw(circleTop);
        g2.draw(circleLeft);
        g2.draw(circleRight);
    }
}
```

Having introduced the additive and subtractive color synthesis systems, our next question is how the RGB system is implemented in a modern digital computer system.

Most color monitors on the market nowadays support the so-called true color, that is 24-bit color. A 32-bit color monitor typically allocates 8 bits for red, 8 bits for green, 8 bits for blue, and 8 bits for the transparency (usually called the *alpha* channel). The alpha channel indicates how transparent the graphical element is to which the color is assigned, when overlaid on other elements, to be discussed in more details in Sect. 7.5. The 8 bits for each of red, green, and blue gives 256 different color shades. The combined 24-bit RGB true color space gives 2^{24} or 16,777,216 different color variations.

The required storage for a high resolution system is fairly high. For example, a 1024×1024 image at 32-bit depth would require 4 MB of storage, calculated as follows: 1024 pixels $\times$ 1024 pixels $=$ 1048576 pixels on screen, and each pixel uses 32 bits (=4 bytes) for color coding, making the total number of bytes needed to represent an image $4 \times 1048576 = 4$ MB.

Although 16 millions of different color variations sound sufficient for any application, they are in fact not when it comes to color gradients. The 24-bit true color space may sometimes fail in creating smooth gradients and instead generate bands of color, known as *color banding* effect. Therefore in recent years, the concept of deep color becomes available to support 30/36/48 bits color per pixel for three RGB colors, or 10/12/16 bits per channel. Adding an alpha channel correspondingly makes it 40/48/64 bits per pixel.

With a 64-bit system, everything is doubled. Each of red, green and blue planes is represented by 16 bits, implying a total of $2^{16} = 65{,}536$ levels of color shades for each color. Therefore, the storage for an image with 64-bit per pixel would require 8 MB. The constant drop in memory prices around the world means that the storage requirement would not be a major problem.

In the middle range, a 16-bit color system typically dedicates 5 bits each for red, green, and blue, making a total of $2^{15} = 32{,}768$ colors. The remaining 1 bit is used for transparency.

The lowest resolution, i.e. 8-bit color, system is still widely used, particularly in embedded systems due to its low cost and low storage requirement. Such a system commonly devotes 3 bits for red, 3 bits for green, and 2 bits for blue.

As mentioned earlier, color lookup tables (CLUT) are an alternative method for providing a large number of color options to the user without requiring a large frame buffer. For example, the color lookup table for an 8-bit per pixel display is shown in Fig. 7.4, where each pixel has an 8-bit color index. In this example, each pixel can have any of the 256 distinct colors and each table entry uses 24 bits to specify an RGB color (8 bits for red, 8 bits for green, 8 bits for blue). Figure 7.4 illustrates that a value of 196 stored at the pixel position (x, y) points to the corresponding location in the CLUT. This location stores a hexadecimal value of 0×021031 (binary 0000 0010 0001 0000 0011 0001), which is equivalent to the decimal value of 135217. The binary representation of 0×021031 consists of three 8-bit color codes for RGB, specifying the color intensities (0×02, 0×10 and 0×31, respectively) of the three electron guns in a color monitor.

The original aim of color look up tables (CLUT, Fig. 7.4) was mainly for the economic use of color storage and, as an added benefit, obtaining flexible color ranges. As the storage hardware becomes increasingly cheap, the first objective is no longer that important. Instead, the second benefit is now used by many popular imaging software, such as Photoshop. For example, the last few versions of Photoshop include a tool called LUT (or Look Up Tables). The user could think

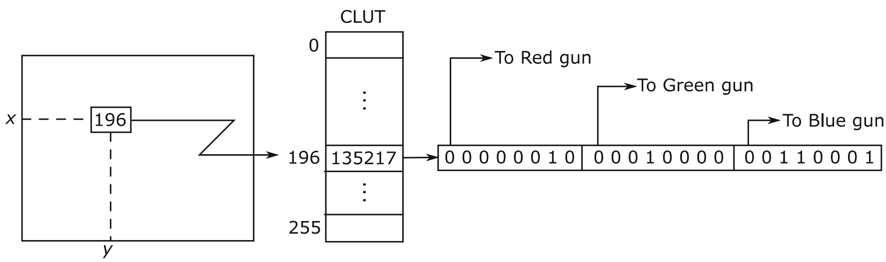


Fig. 7.4 Color look-up table (CLUT)

of LUT as a way to apply a color scheme to a photo, such as mostly reddish colors. This type of technique is often used in the video industry in creating special effects.

Typical RGB input devices include color TV sets, video cameras, digital cameras, image scanners, and game stations. Typical RGB output devices include computer monitors, smartphones, color TV sets (some with color LED displays), video projectors, and large screens.

7.4 HSV and HSL Color Models

Apart from the aforementioned RGB and CMY color models, other popular color models include HSV and HSL models. Here, H is for hue, the prime color of the light; S for saturation or chroma, measuring the purity of a hue, i.e. the percentage of gray; V for value and L for lightness or luminosity, measuring the relative degree of black or white mixed with a given hue. To explain the three terms in real life, we might use hue to tell the difference between a red delicious apple and a gold delicious apple; use saturation to differentiate black coffee from white coffee; and use lightness to differentiate a bread from a toast.

Hue is measured between 0° and 360°, and equals 0° for pure red, 120° for pure green, and 240° for pure blue. With an equal mix of pure red and pure green, hue is 60°, yielding yellow. Similarly, an equal mix of pure green and pure blue produces cyan, for which hue is 180°.

Luminance (or intensity or brightness) equals average of maximal and minimal intensities of red, green, and blue, measuring the overall brightness of light.

Saturation (or chroma) measures relative intensity of the brightest primary color versus least bright primary color in the range of 0 and 1.

The 3-dimensional space representing the HSV model can be easily derived by relating the HSV parameters to the directions in the RGB cube shown in Fig. 7.1a. Viewing the RGB cube along the diagonal from the white vertex toward the black vertex (origin of the 3D space), the outline of the cube is a hexagon as demonstrated in Fig. 7.1b. The six vertices along the outline correspond directly to the six hues in the HSV model. This hexagon forms 360° starting from pure red at 0° counterclockwise. It is used as the top face of a hexcone, where the vertical axis through the center of the hexcone measures the parameter V (value), as shown in Fig. 7.5 on the left. V varies from 0 at the apex of the hexcone, representing black, to 1.0 at the hexagon plane, representing white.

The saturation parameter S varies from 0 at the center V axis, representing the gray level line, to 1.0 at boundary of the hexagon, representing a pure color. If both V and S are non-zero, the hue H varies from 0 to 360°.

The HSL model is similar to HSV model, except it is represented by a double-hexcone, shown in Fig. 7.5 on the right. The luminance parameter L replaces value V as the central axis. The bottom apex represents black where $L = 0$ and the top apex represents white where $L = 1.0$. Pure colors lie on the $L = 0.5$ plane, and thus the grayscale values are along the L axis.

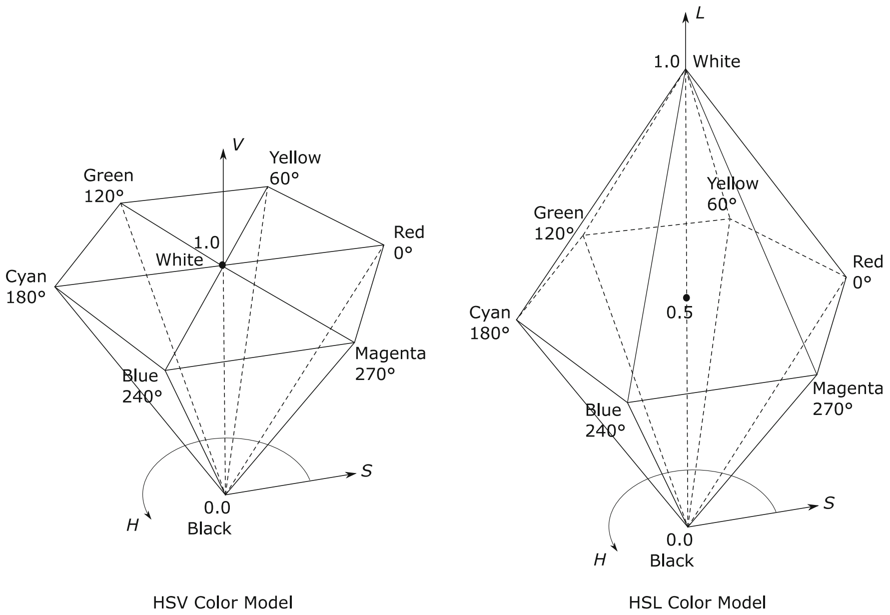


Fig. 7.5 3D representation of HSV and HSL color models

Although more intuitive, both the HSV and HSL models do not correspond to displaying and printing, so to use for such purposes, some kind of conversion method is needed. For both color models, we will now discuss how H, S and V or L can be computed if the values R, G and B are given.

While the given values of R, G and B range from 0 to 255, we prefer ranges from 0 to 1, so we will use

$$r = \frac{R}{255}$$
$$g = \frac{G}{255}$$
$$b = \frac{B}{255}$$

We now compute the largest and the smallest of these three color values and also their difference:

$$c_{\max} = \max(r, g, b)$$
$$c_{\min} = \min(r, g, b)$$
$$\Delta = c_{\max} - c_{\min}$$

Then, for both the HSV and the HSL color models, we can now compute the hue H. First we compute a value h less than 6. If $\Delta = 0, h = 0$. If $\Delta \neq 0$, the value of h depends on which of the r, g and b values is the largest:

$$h = \begin{cases} \dfrac{g-b}{\Delta}, \text{if } c_{\max} = r \\ \dfrac{b-r}{\Delta} + 2, \text{if } c_{\max} = g \\ \dfrac{g-b}{\Delta} + 4, \text{if } c_{\max} = b \end{cases}$$

Then

$$H = \begin{cases} h \times 60^{\circ}, \text{if } h \geq 0 \\ (h+6) \times 60^{\circ}, \text{if } h < 0 \end{cases}$$

The three cases in the computation of h correspond to parts of the horizontal hexagon in Fig. 7.5, each part consisting of two equilateral triangles. For example, if $c_{\max} = g$, the hue will be between yellow and cyan, and, in particular, if then $b = r$, it will be green, with $h = 2$ and $H = 120^{\circ}$.

For the HSV model, we have

$$S = \begin{cases} 0, & \text{if } c_{\max} = 0 \\ \dfrac{\Delta}{c_{\max}}, & \text{if } c_{\max} \neq 0 \end{cases}$$

and

$$V = c_{\max}$$

These values of S and V are expressed as fractions. For example, if $S = 0.75$, this is the same as 75%.

For the HSL model, we compute $c_{\min}$, $c_{\max}$, Δ and H the same way as above. Then we obtain L and S as follows:

$$L = \frac{c_{\min} + c_{\max}}{2}$$

$$S = \begin{cases} 0, \text{if } L = 0 \text{ or } L = 1 \\ \dfrac{\Delta}{2\mathrm{L}}, \text{if } 0 < L \leq 0.5 \\ \dfrac{\Delta}{2 - 2L}, \text{if} 0.5 < L < 1 \end{cases}$$

Again, L and S need to be multiplied by 100% if a percentage is desired.

7.5 Transparency

When two or more colors overlaying on each other, there is a question on whether a later drawn color should completely overwrite the previous color in the overlapping area, or let the underneath color partially or completely show through. This question is like asking whether the top color is transparent or not.

Many computer graphics APIs support the concept of transparency. With transparency as a parameter, color is no longer represented as a RGB triple, but rather a quadruple (*r, g, b, a*), where *a* is the transparency level. To be more precise, "transparency" here should really be called "blending", since the effect of transparency is in fact modeled by using color blending. Color blending compares the color of the pixel to be drawn with the current color of that pixel, and then decides the final color to be used for the pixel based on a predefined blending function. Therefore, in the so-called *RGBA model*, any color object also has a color blending property, represented by translucent pixels. Instead of simply associating 1 extra bit of data with each pixel to simply indicate it being either completely transparent or not, the graphics system can associate 4, 8, 16, or some other number of bits with each pixel. This leads to 16, 256, or 65,536 possible levels of translucency, ranging from fully transparent (0) to fully opaque (16, 256, or 65,536). Recall Sect. 7.3 on the RGB representation, these transparency bits are really no different from the bits used to represent the red, green, and blue components of each pixel.

The transparency bits measure the blending level, and make the *a* parameter in the above quadruple (*r, g, b, a*), called the *alpha value*, set between 0.0 and 1.0, determining opacity. Assuming a standard blending function used, if the alpha value is 1.0, the new color to be drawn is completely opaque, implying that the existing color is overwritten by the new color. Similarly, if the alpha value is 0.0, the new color is completely transparent so that it is completely taken over by the existing color. With an alpha value between 0.0 and 1.0, the new color is combined with the existing color based on the blending function to produce a transparency effect.

In Java 2D, one can assign alpha values to drawing operations so that the underlying graphics partially shows through shapes or images. This is done by setting a transparency level by creating an *AlphaComposite* object and then passing the *AlphaComposite* object to the *setComposite* method of the *Graphics2D* object. To create an *AlphaComposite* object, we make a call to *Alpha-Composite.getInstance* with two parameters, i.e. the composing rule and an alpha value. Of the eight compositing rules for the *AlphaComposite* API, *SRC_OVER* is the most commonly used. The *SRC_OVER* compositing rule places the source color over the destination color and blends them based on the transparency of the source, using the formula that will be discussed later in this section.

The following program demonstrates the use of the *AlphaComposite* API for 11 levels of transparency from being fully transparent to fully opaque, i.e. alpha value from 0.0 to 1.0 with 10 increments of 0.1. The resulting transparency effects are shown in Fig. 7.6, where a blue circle is partially overlapping a red square.

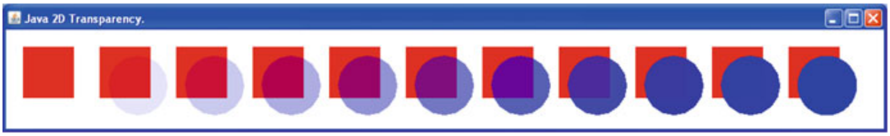


Fig. 7.6 Different levels of transparency

```
// Transparency.java: draws 11 blue circles over 11 red squares,
// with transparency alpha changed from 0.0 to 1.0
import java.awt.*;
import java.awt.event.*;
import java.awt.geom.*;

public class Transparency extends Frame {
   public static void main(String[] args){new Transparency();}
   Transparency() {
      super("Java 2D Transparency.");
      addWindowListener(new WindowAdapter()
        {public void windowClosing(WindowEvent e){System.exit(0);}});
            setSize(1040, 150);
      add("Center", new CvTransparency());
      setVisible(true);
      }
}

class CvTransparency extends Canvas {
    private static int gap=20,width=60,offset=10,deltaX=gap+width
    +offset;
    private Ellipse2D blueCircle = new Ellipse2D.Double(gap+offset,
       gap+offset, width+offset, width+offset);
   private Rectangle redSquare = new Rectangle(gap, gap, width, width);

   private AlphaComposite makeComposite(float alpha) {
      int type = AlphaComposite.SRC_OVER;
      return(AlphaComposite.getInstance(type, alpha));
   }
   private void drawShapes(Graphics2D g2, float alpha) {
      Composite myComposite = g2.getComposite();
      g2.setPaint(Color.red);
      g2.fill(redSquare);
      g2.setComposite(makeComposite(alpha));
      g2.setPaint(Color.blue);
      g2.fill(blueCircle);
      g2.setComposite(myComposite);
   }
```

```
    public void paint(Graphics g) {
        super.paint(g);
        Graphics2D g2 = (Graphics2D)g;
        for(int i=0; i<11; i++) {
            drawShapes(g2, i*0.1F);
            g2.translate(deltaX, 0);
        }
    }
}
```

Returning to the concept of transparency modeled by blending, in reality, there is no such a thing as a translucent pixel. On a computer display, a pixel is either on or off; and cannot be partially on. In order to give the appearance of transparency, the graphics system composite (blend) transparent pixels with the pixels that are beneath them. If a source color C_s has a transparency level defined by the alpha value α, when C_s is painted over a destination color C_d, the two colors are combined to produce a new color C. The combination is defined by some simple mathematical formula, such as the following:

$$C = C_s\alpha + C_d(1 - \alpha)$$

If C_s is fully opaque, α is 1 and C is simply the source color C_s. On the other hand, if C_s is fully transparent, α is 0 and C is simply the destination color C_d. If α is somewhere between fully opaque and fully transparent, C is a combination of the source and destination colors. The computation is performed independently on each of the red, green, and blue colors. Such a simple mathematical formula can effectively produce blended colors that appear visually convincing, just like translucent colors.

7.6 Texture

To create realistic images, one can add texture onto a 2D polygon, or the surface of a 3D object. Texture mapping can be considered another way of defining colors to fill a polygon. This process is similar to applying patterned paper to a plain white wall when decorating a house.

The color of each pixel in the applied polygon is calculated by including information from an array of values called a *texture map*. The texture map can be 1-dimensional, 2-dimensional or 3-dimensional. For example, a 1-D texture may be used to create a pattern for coloring a line. A 3D texture may specify a solid block of a 3D pattern which could be used to sculpt a 3D object. Mapping 2D textures to surfaces finds the most applications and is therefore a widely studied topic. We therefore focus on 2D texturing in this section.

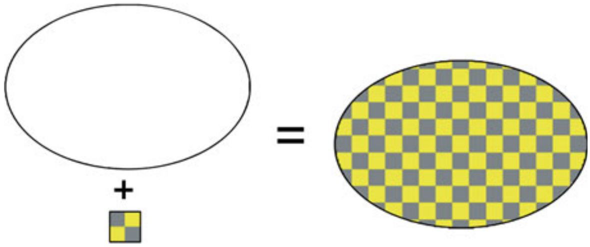


Fig. 7.7 Texture mapping

Conceptually there are two spaces in textural mapping: the usual 2D screen space in which graphics objects are displayed, and the texture space that keeps the texture map, as illustrated in Fig. 7.7. When a texture map (bottom left in Fig. 7.7) is properly pasted onto an object in the 2D screen space (top left), a textured object is created (the textured ellipse on the right). The two spaces are realized in two graphics contexts in Java (e.g. *gi* and *g*2 in *Texture.java* below). The key to a proper texture mapping is to establish a link and precise alignment between the two spaces.

To determine the texture color for a texture mapped pixel, *texture filtering* is performed using the colors of nearby texture elements, which are usually called *textels*. It is called textels rather than pixel in order to emphasize its role in texture filtering. Since the textured surface may be at an arbitrary distance and orientation relative to the viewer, one pixel does not usually correspond directly to one textel. Some form of filtering has to be applied to determine the best color for the pixel.

When more than one texture is used at a time on a polygon or surface, it is called *multi-texturing*. For example, to avoid recalculating that lighting every time a surface is rendered, a texture known as light map may be used to light the surface. Bump mapping is another useful multi-texture technique, that could generate realistic image effects for complex surfaces. A bump map distorts the normal vector during the shading process to draw the surface with small variations in shape, such as rough concrete. It shows lighting details in addition to the usual texture coloring. As the modern GPU technology has become increasingly more powerful, complex bump mapping can now be generated in real time and thus is often used in computer games.

The following program generates three images as shown in Fig. 7.8:

– A vertical stripe of repeated color gradient from yellow to blue;
– A text string "TEXTURE" applied with a generated texture image of 6×6 pixels (gray background with a red angle); and
– A rectangle filled with a texture defined by an image file.

In this example, a rectangle of 700×220 pixels is filled with repeated images, each of 50×55 pixels given in the image file *mondrian.png*, so in Fig. 7.8 this image occurs 14×4 times.

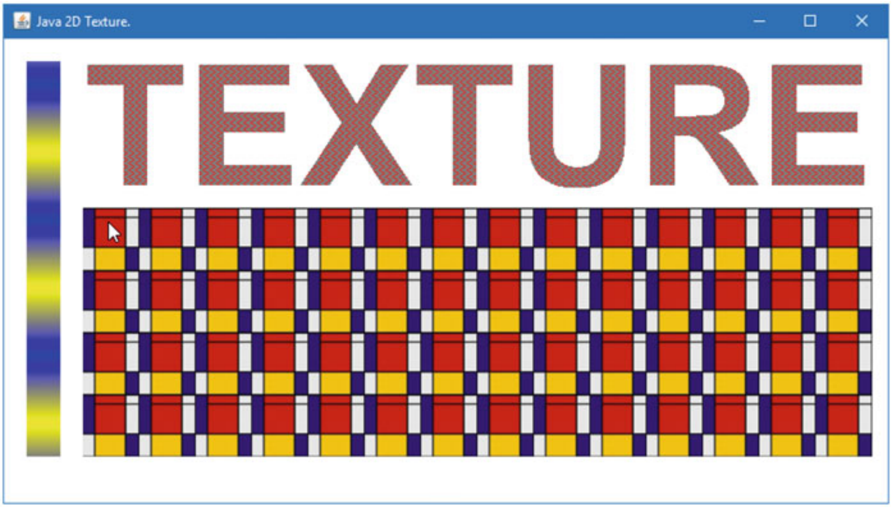


Fig. 7.8 Examples of color gradient and texture mapping

```
// Texture.java: draw (1) a vertical strip of repeated color gradient,
// (2) a texture created using BufferedImagepaint, and applied on a
// text string "TEXTURE", and (3) a texture defined by an image file.

import java.awt.*;
import java.awt.event.*;
import java.awt.geom.*;
import java.awt.image.*;
import java.io.*;
import java.net.URL;
import javax.imageio.*;

public class Texture extends Frame {
   public static void main(String[] args){new Texture();}
   Texture() {
      super("Java 2D Texture.");
      addWindowListener(new WindowAdapter()
        {public void windowClosing(WindowEvent e){System.exit(0);}});
      setSize(800, 450);
      add("Center", new CvTexture());
      setVisible(true);
   }
}
```

```
class CvTexture extends Canvas {
   private BufferedImage image;
   public void paint(Graphics g) {
      super.paint(g);
      Graphics2D g2 = (Graphics2D)g;
      // Gradient Strip
      GradientPaint gp =
        new GradientPaint(20,100,Color.yellow,20,160,Color.blue,true);
      g2.setPaint(gp);
      g2.fillRect(20, 20, 30, 350);
      // Generating texture of 6x6 pixels
      image = new BufferedImage(6, 6, BufferedImage.TYPE_INT_RGB);
      Graphics2D gi = image.createGraphics();
      gi.setColor(Color.gray);
      gi.fillRect(0,0,6,6);
      gi.setColor(Color.red);
      gi.drawLine(0,0,6,3);
      gi.drawLine(0,6,6,3);
      TexturePaint tp = new TexturePaint(image,
         new Rectangle(50,20,6,6));
      g2.setPaint(tp);
      Font f = new Font("Arial", Font.BOLD, 150);
      g2.setFont(f);
      g2.drawString("TEXTURE", 70, 130);
      // Image file as texture
      URL url=getClass().getClassLoader().getResource("mondrian.png");
      try {
        image = ImageIO.read(url);
        } catch (IOException ex) {
        ex.printStackTrace();
        }
      tp = new TexturePaint(image, new Rectangle2D.Double(70,150,
          image.getWidth(), image.getHeight()));
      g2.setPaint(tp);
      Shape rectangle = new Rectangle(70,150,700,220);
      g2.fill(rectangle);
   }
}
```

7.7 Surface Shading

To make a 3D object appear to have volume and to be photorealistic, we could use shading, by painting the surface with light. In computer graphics, shading refers to the process of determining the color of a face (polygon) of a 3D object to create a photorealistic effect, based on the face’s angle to lights and distance from lights.

When light casts onto the surface of an object, it could be absorbed, reflected or refracted. The absorbed light is transformed into the heat, while reflected and refracted light are visible to the human eye so that the object could be seen. To model such effects, we build mathematical models rather than complex physical models, which are generally called the *shading models*. The models to be discussed in this section are all the so-called "local" models in the sense that they do not consider secondary reflections. In other words, light reflected from several surfaces before reaching the eye is not considered.

Based on the principals of optical physics, a shading model determines how to color the surface of a given object. When light shines on an object surface, there are three components. First, the light can be reflected through the object surface to the space, resulting in *reflection*. Secondly, for a transparent body, the light can penetrate the object and come out from the other side, resulting in *refraction*. Finally, part of the light is absorbed by the object surface and converted into heat. Among these three components, only the reflection and refraction can enter the human eye to produce a visual effect.

Reflection and refraction on the object surface determine the color of the object. Specifically, the intensity of reflection and refraction determines the brightness of the surface of the object. Reflection and refraction contain different wavelengths of light that determine the surface color of the object, which in turn depends on the incident light and the degree of absorption of light at different wavelengths. For example, when a beam of white light is illuminated on a surface of an opaque object that absorbs any wavelengths but the red wavelength, the surface of the object is red. However, if a beam of green light illuminates the object, the object appears black. Therefore, without light, it makes no sense to discuss the color of an object.

Therefore, to accurately calculate the surface of the reflection and refraction, the need to know the incident light intensity at each wavelength and the surface of the object on each wavelength of light absorption rate, that is, to understand the spectral distribution of incident light and the reflectivity of the surface of the object and the transmittance. This is related to the nature, shape, quantity and position of the light source, but also to many factors such as geometric shape, optical property, surface texture, and even physiological and psychological visual factors. It is not possible to calculate all these accurately, so we need to find out the main factors, establish the mathematical model, and choose the appropriate model according to the application.

Simple shading models assume point light sources, opaque objects with smooth surface, so that refraction and scattered ambient light are negligible. The part being absorbed by the object would not generate any visual effect, hence simple shading models only consider the effect of reflection.

Flat Shading

Flat shading is a shading model where the faces (polygons) of a 3D object have no corrective algorithm for reflection of light. Therefore, each of the faces reflects as a flat surface, giving a blocky look and feel to the model, and thus known as *flat shading*. Comparing with more complex shading models, such as Gouraud, Phong, Blinn-Phong, or Lambert, flat shading requires relatively little computational overhead and thus runs fast. It therefore generates least realistic looking 3D objects.

We now use the simplest flat shading model to color each polygon (and for all triangles of which it consists) based on its orientation in relation to the light source. Let us consider a polygon in a plane with equation

$$ax + by + cz = h,$$

where $N = (a, b, c)$, perpendicular to that plane, is normalized (that is, $a^2 + b^2 + c^2 = 1$). We can then obtain a color code by computing the dot product of $\boldsymbol{N}$ and the normalized light vector $\boldsymbol{L}$. The latter is a vector (also of length 1) pointing from the object to the sun (that is, to the light source, assumed to be infinitely far away). Expressed in eye coordinates, we will use

$$\boldsymbol{L} = \left(1/\sqrt{3}\right)(-1, 1, 1)$$

Recall that in the eye coordinate system the x-axis points from the object toward us, so the minus sign in this choice of $\boldsymbol{L}$ means that the sun's position is in front of the object. Clearly, the higher the value of the dot product $(\boldsymbol{N} \cdot \boldsymbol{L})$, the brighter the polygon should be displayed. The vectors $\boldsymbol{N}$ and $\boldsymbol{L}$ are shown in Fig. 7.9.

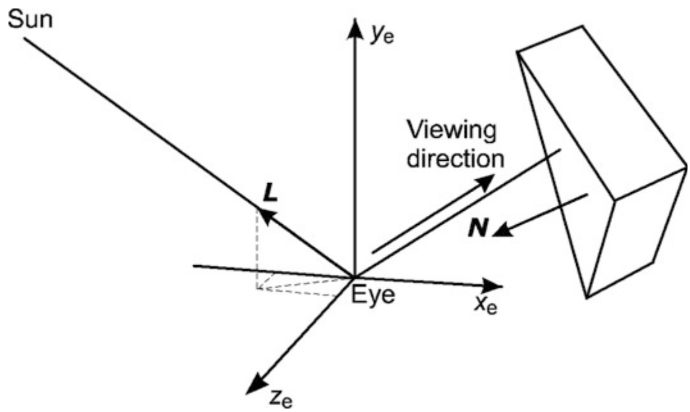


Fig. 7.9 Vector **L**, pointing to the sun, and normal vector **N** of a face

Ambient Light

Without any source of light, an object would still be visible as its surface has a bit of shade. This could be considered an indirect result of many light sources, having been reflected many times and diffused around the space, known as *ambient light*. A major characteristic of ambient light is its omni-directional, with a fixed intensity and color which equally affect all objects in the scene.

We can use a constant to simulate ambient light, using the formula:

$$I_A = L_A M_A$$

where L_A is the intensity of ambient light and M_A is the object's ambient reflection coefficient, ranging between 0.0–1.0 for each of R, G, and B, determined by the material property of the object. Under the same ambient light, the light intensity of the surface of an object may not be the same, and thus is modeled by M_A.

Diffuse Shading and Lambertian Model

Under many circumstances, light has a direction, such as sunlight. Given a relatively rough surface, light is reflected from the surface in all directions in equal intensity or luminance. This type of reflection in many angles is called *diffuse reflection*, in contrast to *specular reflection* at a single angle (see Phong Shading next).

The intensity of diffuse reflection could be approximated by the Lambertian reflection that models an ideal diffusely reflecting surface. A Lambertian surface should have the same perceived brightness and color from all viewing directions if the surface is uniformly lit. Such a surface obeys Lambert's cosine law, which states that the intensity of the diffuse reflection on a surface is proportional to the cosine of the angle between the direction of light and the surface's normal vector.

Assume that I_d is the intensity of the light hitting the surface, $\boldsymbol{N}$ is the normal vector at point P of the surface, and θ is the angle between $\boldsymbol{N}$ and $\boldsymbol{L}$ (as shown in Fig. 7.10). According to Lambert's Cosine Law, the intensity of diffuse reflection at P is given by:

$$I_{diffuse} = I_d K_d \cos\theta = I_d K_d (\boldsymbol{L} \cdot \boldsymbol{N})$$

where $K_d (0 < K_d < 1)$ is the diffuse reflection coefficient. This formula implies that the intensity of diffuse reflection is only influenced by the angle of incoming light and has no relation to the angle of reflection, i.e. the viewpoint's position.

The intensity is the highest if the normal vector points in the same direction as the light vector (i.e. $\cos 0 = 1$), the surface is perpendicular to the light's direction). It is the lowest if the normal vector is perpendicular to the light vector (i.e. $\cos 90^{\circ} = 0$),

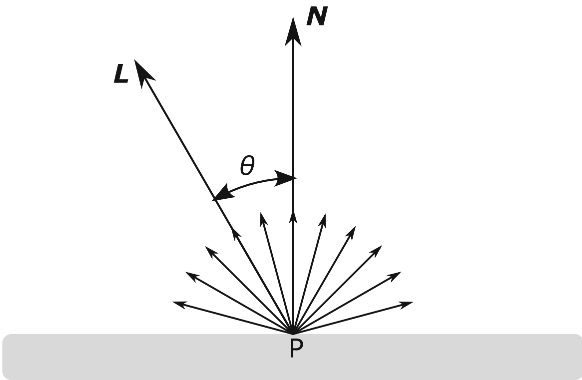


Fig. 7.10 Diffuse reflection

the surface is in parallel with the light's direction). If $\cos\theta < 0$, the light source comes from the backside of the surface, and thus does not reach the point P.

In a RGB color system, the diffuse reflection coefficient K_d has three components, K_{dr}, K_{dg}, $K_{db,}$ corresponding to the coefficients for red, green, and blue respectively. The three coefficient components together represent the color of the surface. Similarly, we could also divide I_d into three components I_r,I_g, I_b and color the surface by assigning the three values.

Phong Lighting Model

The above Lambertian reflection works well for rough surfaces, such as wall and paper, but does not work for metal surfaces that may shine under a light source, since it does not consider reflection from the surface.

When light casts on a smooth surface, there is a specular reflection, commonly known as highlight. Assume again $\boldsymbol{N}$ to be the normal vector at point P of the surface, while $\phi(0 \le \phi \le \pi/2)$ is the angle between $\boldsymbol{N}$ and the vector $\boldsymbol{V}$ pointing toward the viewer. The light ray $(-\boldsymbol{L})$c reflects at the other side ($\boldsymbol{R}$) of $\boldsymbol{N}$ at the same angle θ (Fig. 7.11).

To compute the vector $\boldsymbol{R}$, we consider the rhombus shown in Fig. 7.12, a parallelogram with four sides of length 1. Since both the light vector $\boldsymbol{L}$ and the normal vector $\boldsymbol{N}$ have length 1, the projection of $\boldsymbol{L}$ on $\boldsymbol{N}$ is equal to $L \cdot N = \cos\theta$, and the vector sum $\boldsymbol{L}+\boldsymbol{R}$ has length $2(\boldsymbol{L}\cdot\boldsymbol{N})$. Therefore we can compute

$$\boldsymbol{R} = 2(\boldsymbol{L}\cdot\boldsymbol{N})\boldsymbol{N} - \boldsymbol{L}$$

The Phong lighting model computes the specular reflection using the following formula:

$$I_{spec} = I_d K_s(\cos\varphi)^n = I_d K_s(\boldsymbol{V}\cdot\boldsymbol{R})^n$$

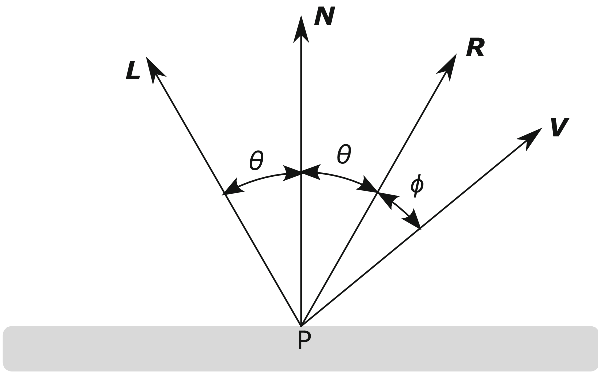


Fig. 7.11 Specular reflection

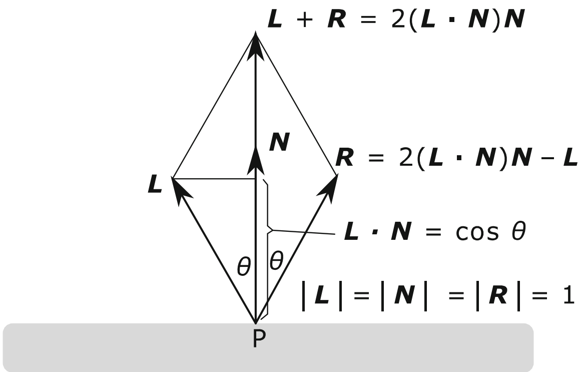


Fig. 7.12 Light, normal and reflection vectors

where I_d is the intensity of the light hitting the surface, K_s is the specular reflection coefficient, and n is the Phong exponent. This formula implies that specular reflection depends on the viewing direction, as well as the intensity of incident light. Given I_d and K_s, when the direction of the viewpoint $\boldsymbol{V}$ lines up with the reflection $\boldsymbol{R}$, i.e. $\phi = 0$, the highlight is the strongest. The highlight falls off gradually when the viewpoint $\boldsymbol{V}$ moves away from $\boldsymbol{R}$.

Adding all the above three lights together, i.e. ambient, diffuse, and specular, we can obtain a more realistic looking scene under the Phong lighting model:

$$I = I_{ambient} + I_{diffuse} + I_{specular}$$

where

$$\begin{aligned}
I_{ambient} &= I_a K_a \\
I_{diffuse} &= I_d K_d \cos\theta = I_d K_d (\boldsymbol{L} \cdot \boldsymbol{N}) \\
I_{specular} &= I_d K_s (\cos\phi)^n = I_d K_s (\boldsymbol{V} \cdot \boldsymbol{R})^n
\end{aligned}$$

The Phong lighting model is the simplest and also the most popular lighting and shading model in 3D computer graphics. It is flexible in achieving a wide range of visual effects and easy to implement in both software and hardware. In fact, almost all the graphics hardware and game stations implement the Phong shading model.

Java Example

The class *Obj3D*, listed in Appendix B, contains the implementation of the above theory. The following method of this class computes the illumination of a polygon. It depends only on the normal vector of the polygon, so the x, y and z components of this vector are supplied as arguments. Since we are using eye coordinates (see Fig. 5.4) and the viewing vector $\boldsymbol{V}$ points from the object towards the viewpoint, we have $V = (0, 0, 1)$. We choose the light vector, also in eye coordinates, as

$$\boldsymbol{L} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Since, with eye-coordinates, the x-axis points to the right, the y-axis upward and the z-axis towards the viewer, this means that the light comes more or less from the left, from above and from the front.

As for the constants K_a, K_d andK_s, we will set diffuse illumination (with $K_s = 0$) by default, while the user can use the *View* menu to switch to specular illumination (with $K_s = 0.2$). For this purpose, the class *Obj3D* also contains the following method:

```
void setSpecular(Boolean isSpecular) {
   if (isSpecular) {
      kAmb = 0.2; kDiff = 0.7; kSpec = 0.2;
   }
   else {// Diffuse
      kAmb = 0.4; kDiff = 0.6; kSpec = 0.0;
   }
}
```

Figure 7.13 shows the default diffuse illumination while the *View* menu shows how to switch to specular illumination.

After using this command, Fig. 7.14 appears; although not shown here, in this state the *View* menu enables the user to switch to diffuse illumination.

The code that actually computes the colors for all polygons is shown in the method *colorCodePhong* below.

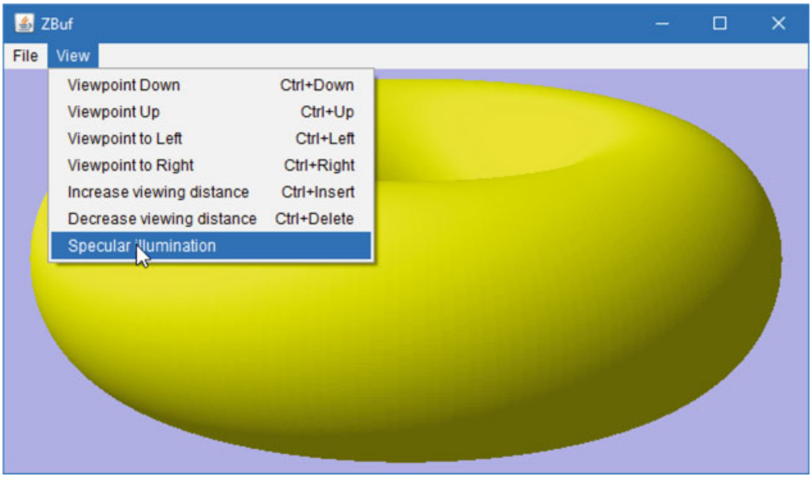


Fig. 7.13 Diffuse illumination

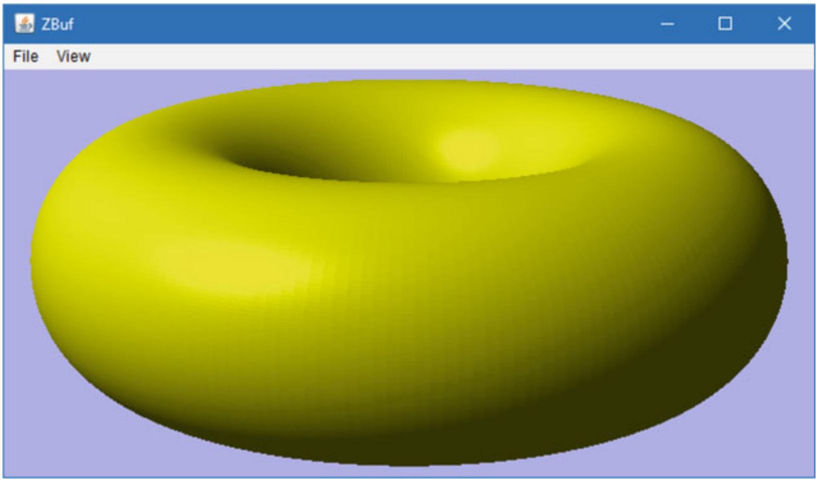


Fig. 7.14 Specular illumination (specular component set non-zero)

```
int colorCodePhong(double xN, double yN, double zN) {
  // Viewing vector V (from O to E, length 1):
  double colorAmbR = 1, colorAmbG = 1, colorAmbB = 0,
         colorDifR = 1, colorDifG = 1, colorDifB = 0,
         colorSpecR = 1, colorSpecG = 1, colorSpecB = 0;
  // Red (R) and green (G) without blue (B) gives yellow.
  // Ambient component:
```

```
  double illumAmbR = kAmb * colorAmbR,
         illumAmbG = kAmb * colorAmbG,
         illumAmbB = kAmb * colorAmbB;

  // Diffuse component:
  double
     inprodLN = Math.max(0, xL * xN + yL * yN + zL * zN),
     illumDiff = inprodLN * kDiff,
     illumDiffR = illumDiff * colorDifR,
     illumDiffG = illumDiff * colorDifG,
     illumDiffB = illumDiff * colorDifB;

  // Specular component:
  // Reflection vector R = 2(L . N)N - L
  // xR and yR would only be used to multiply them by xV
  // and yV, and these are zero since V points to the
  // viewpoint E and we are using eye coordinates, so
  // computing xR and yR would be useless.
  double zR = 2 * inprodLN * zN - zL, // // xV = yV = 0:
         dotProductVR = Math.max(0, zV * zR),
         illumSpec = kSpec * Math.pow(dotProductVR, 16),
         illumSpecR = illumSpec * colorSpecR,
         illumSpecG = illumSpec * colorSpecG,
         illumSpecB = illumSpec * colorSpecB;

  // Sum of ambient, diffuse and specular illumination:
  double
     illumR =
         Math.min(1, illumAmbR + illumDiffR + illumSpecR),
     illumG =
         Math.min(1, illumAmbG + illumDiffG + illumSpecG),
     illumB =
         Math.min(1, illumAmbB + illumDiffB + illumSpecB);

  int red = (int) (255 * illumR),
      green = (int) (255 * illumG),
      blue = (int) (255 * illumB);
  return (red << 16) | (green << 8) | blue;
  }
}
```

Near the top you can see that the red and green color components are set to 1 and the blue ones to 0, since we want to display the object in shades of yellow. This color-coding is easy to understand; the program is also easy to modify if any color other than the current shade of yellow is desired.

Exercises

7.1. Program *ColorCircles.java* in Sect. 7.3 is based on the additive color system. Modify this program so that it produces the diagram for the subtractive color system as shown in Fig. 7.2b,

7.2. Write a program to fill every face of the hexcone for the HSV model of Fig. 7.5. For each triangle, use the RGB values of its vertices to find the RGB value of each pixel in the triangle by interpolation.

7.3. Draw three partially overlapping copies of capital letter A to display various effects of transparency, gradient, and texture.

7.4. Modify class *Obj3D* to change the appearance of the images with regard to colors and the ambient, diffuse and specular illumination components.

Chapter 8
Fractals

There are many aspects in nature that are repeating and in many cases in patterns similar at different scales. For example, when observing a pine tree, one may notice that the shape of a branch is very similar to that of the entire tree, and the shapes of sub-branches and the main branch are also similar. Such kind of self-similar structure that occurs at different levels of magnification can be modeled by a branch of mathematics called *Fractal geometry*. The term *fractal* was coined by Benoît Mandelbrot in 1975, and means *fractus* or *broken* in Latin. Fractal geometry studies the properties and behavior of fractals. It describes many situations which cannot be explained easily by classical geometry. Fractals can be used to model plants, weather, fluid flow, geologic activity, planetary orbits, human body rhythms, socioeconomic patterns, and music, just to name a few. They have been applied in science, technology, and computer generated art. For example, engineers have been using fractals to control fluid dynamics in order to reduce process size and energy use.

A fractal, typically expressed as curves, can be generated by a computer recursively or iteratively with a repeating pattern. Compared with human beings, computers are much better in processing long and repetitive information without complaint. Fractals are therefore particularly suitable for computer processing. This chapter introduces the basic concepts and program implementation of fractals. It starts with a simple type of fractal curves, then focuses on a grammar-based generic approach for generating different types of fractal images, and finally discusses the well-known Mandelbrot set.

8.1 Koch Curves

A simple example of self-similar curves is the Koch curve, discovered by the Swedish mathematician Helge von Koch in 1904. It serves a useful introduction to the concepts of fractal curves.

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2_8

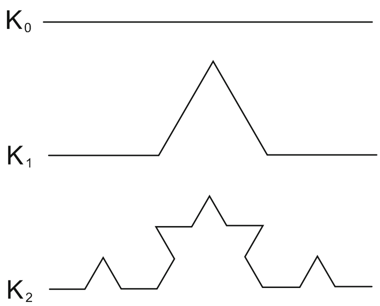


Fig. 8.1 Three generations of the Koch curve

In Fig. 8.1, K_0, K_1, and K_2 denote successive generations of the Koch curve. A straight line segment is called the zero-th generation. We can construct Koch curve as follows:

- *Begin with a straight line and call it* K_0*;*
- *Divide each segment of* K_n *into three equal parts; and*
- *Replace the middle part by the two sides of an equilateral triangle of the same length as the part being removed.*

The last step ensures that every straight line segment of K_n becomes the shape of K_1 in a smaller scale in K_{n+1}.

Koch curves have the following interesting characteristics:

- *Each segment is increased in length by a factor of 4/3. Therefore,* $\mathrm{K}_{\mathrm{n}+1}$ *is 4/3 as long as* K_n*, and* K_i *has the total length of* $(4/3)^{\mathrm{i}}$.
- *When* n *is getting large, the curve still appears to have the same shape and roughness.*
- *When* n *becomes infinite, the curve has an infinite length, while occupying a finite region in the plane.*

The Koch curve can be easily implemented using the turtle graphics method. Originated in the Logo programming language, turtle graphics is a means of computer drawing using the concept of a turtle crawling over the drawing space with a pen attached to its underside. The drawing is always relative to the current position and direction of the turtle. Considering each straight line of K_{n-1} to be drawn as a K_1 in the next generation, we can write a recursive program to draw Koch curves as in the following pseudocode:

To draw K_n we proceed as follows:

```
If  (n == 0) Draw a straight line;
Else
{  Draw Kn−1;
   Turn left by 60°;
   Draw Kn−1;
   Turn right by 120°;
```

```
   Draw Kn−1;
   Turn left by 60°;
      Draw Kn−1;
   }
```

To implement the above pseudocode in a Java program, we need to keep track of the turtle's current position and direction. The following program draws the Koch curve at the zero-th generation K_0. With each mouse click, it draws a higher generation replacing the previous generation. The program defines the origin of the coordinate system at the center of the screen. The turtle starts pointing to the right, and locating at the half of the initial length to the left of the origin, that is, $(-200, 0)$ for the initial length of 400. The turtle moves from the current position at (x, y) to the next position at $(x1, y1)$ while changing its direction accordingly.

```
//Koch.java: Koch curves.
import java.awt.*;
import java.awt.event.*;

public class Koch extends Frame {
   public static void main(String[] args) {new Koch();}

   Koch() {
      super("Koch. Click the mouse button to increase the level");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(600, 500);
      add("Center", new CvKoch());
      setVisible(true);
   }
}

class CvKoch extends Canvas {
   public float x, y;
   double dir;
   int midX, midY, level = 1;

   int iX(float x) {return Math.round(midX + x);}
   int iY(float y) {return Math.round(midY - y);}

   CvKoch() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
```

```
            level++; // Each mouse click increases the level by 1.
            repaint();
         }
      });
   }

   public void paint(Graphics g) {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1,
          length = 3 * maxX / 4;
      midX = maxX / 2; midY = maxY / 2;
      x = (float) (-length / 2); // Start point
      y = 0;
      dir = 0;
      drawKoch(g, length, level);
   }

   public void drawKoch(Graphics g, double len, int n) {
      if (n == 0) {
         double dirRad, xInc, yInc;
         dirRad = dir * Math.PI / 180;
         xInc = len * Math.cos(dirRad); // x increment
         yInc = len * Math.sin(dirRad); // y increment
         float x1 = x + (float) xInc, y1 = y + (float) yInc;
         g.drawLine(iX(x), iY(y), iX(x1), iY(y1));
         x = x1; y = y1;
      } else {
         drawKoch(g, len /= 3, --n);
         dir += 60;
         drawKoch(g, len, n);
         dir -= 120;
         drawKoch(g, len, n);
         dir += 60;
         drawKoch(g, len, n);
      }
   }
}
```

Joining three Koch curves together, we obtain the interesting Koch snowflake as shown in Fig. 8.2. The length of a Koch Snowflake is $3 \times (4/3)^i$ for the ith generation since the length of K_i is $(4/3)^i$. It increases infinitely as does i. The area of the Koch snowflake grows slowly and is indeed bounded. In fact, as i becomes very large, its shape and roughness appear to remain the same. Koch snowflakes can be easily drawn by connecting three Koch curves using a modified version of the program above (see Exercise 8.2).

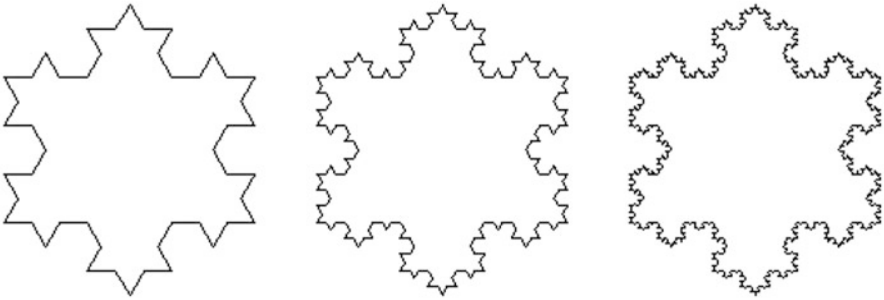


Fig. 8.2 Koch snowflakes of generations 2, 3 and 4

8.2 String Grammars

As discussed above, Koch curves are drawn through a set of commands specifically defined for Koch curves. There are many interesting curves that could be drawn in a similar fashion but would require a complete program for each different kind of curve. The approach in the above section is apparently not general for generating different kinds of curves.

Consider again the Koch curve. The pattern of the first generation repeats itself in smaller scales at higher generations.

Such a common pattern distinguishes Koch curves from other curves. Therefore, an approach that can encode the common pattern in a simple string of characters would be general enough to specify a variety of curves. Formally, the specification of a common pattern is called a *grammar*, and the grammar-based systems for drawing fractal curves are called *L-Systems* (invented by the Hungarian biologist Aristid Lindenmayer in 1968).

The string of characters defining a common pattern instructs the turtle to draw the pattern. Each character in the string serves as a command to perform an atomic operation. Given the distance D and turning angle α in which the turtle is supposed to move, let us now introduce three most common character commands:

- F - *move forward the distance D while drawing in the current direction.*
- $+$ - *turn right through the angle* α.
- $-$ - *turn left through the angle* α.

For example, given a string $F - F + +F - F$ and angle 60°, the turtle would draw the first generation of Koch curve K_1 as shown in Fig. 8.1. It would however be tedious and error-prone to manually provide long strings for different curves. Fortunately, computers are best at performing repeated, long and tedious tasks without making mistakes. Using the same Koch curve example, to draw more generations, we define a *string production rule*

$$F \rightarrow F - F + +F - F$$

The rule means that every occurrence of F (that is, the left hand side of '$\rightarrow$') should be replaced by $F - F + + F - F$ (that is, the right hand side). Starting from an initial string F which is called the *axiom*, recursively applying this production rule would produce strings of increasing lengths. Interpreting any of the strings, the turtle would draw a corresponding generation of the Koch curve. Let us make the axiom the zero-th generation $K_0 = F$ and the first generation $K_1 = F - F + + F - F$, then K_2 can be obtained by substituting every F character in K_1 by $F - F + + F - F$, so that

$$K_2 = F - F + +F - F - F - F + +F - F + +F - F + +F - F - F - F + +F - F$$

By interpreting this string, the turtle would draw the curve exactly the same as K_2 in Fig. 8.1. This process can continue to generate the Koch curve at any higher generations.

In summary, to draw a fractal curve, such as the Koch curve, at any generation, we need to know at least the following three parameters:

1. The axiom from which the turtle starts.
2. The production rule for producing strings from F character. We will call the right hand side of this rule the *F-string*, which is sufficient to represent the rule.
3. The angle at which the turtle should turn.

Denoting these parameters in a template form (axiom, F-string, angle), we call this the *grammar* of the curve, and we can specify the Koch curve as $(F, F - F + + F - F, 60)$.

To define more complex and interesting curves, we introduce three more production rules, obtaining grammars of six elements instead of three as above. We introduce an *X-string*, to be used to replace every occurrence of X when producing the next generation string. Similarly, there will be a *Y-string*, used to replace every occurrence of Y. Their replacement process is performed in the same fashion as with the F-string. In other words, all the three string types are treated equally during the string production process. The X and Y characters are, however, different from the F character as they are simply ignored by the turtle when drawing the curve. The third new production rule, named *f-string*, will be discussed shortly. In the meantime, we reserve its position, but substitute *nil* for it to indicate that we do not use it. It follows that there are six parameters in the extended grammar template

$$(\text{axiom, F-string, f-string, X-string, Y-string, angle}).$$

The following grammars produce some more interesting curves:

Dragon curve: $(X, F, nil, X + YF+, -FX - Y, 90)$.
Hilbert curve: $(X, F, nil, -YF + XFX + FY-, +XF - YFY - FX+, 90)$.
Sierpinski arrowhead: $(YF, F, nil, YF + XF + Y, XF - YF - X, 60)$.

Figures 8.3, 8.4 and 8.5 illustrate some selected generations of the Dragon, Hilbert and Sierpinski curves that are generated based on their string grammars as defined above.

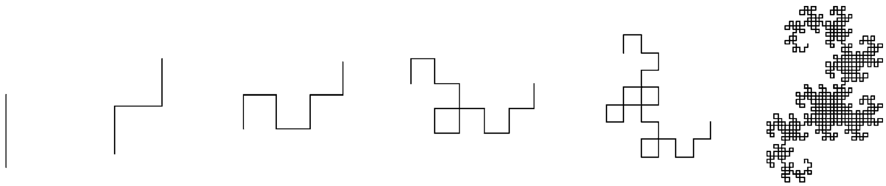
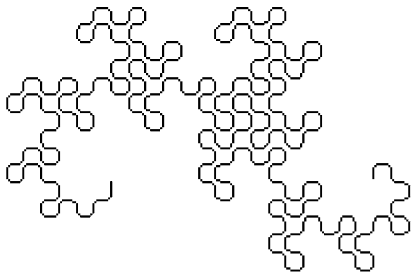


Fig. 8.3 Dragon curves: 1st, 2nd, 3rd, 4th, 5th and 11th generations

Fig. 8.4 Dragon curve of 8th generation, rounded corners



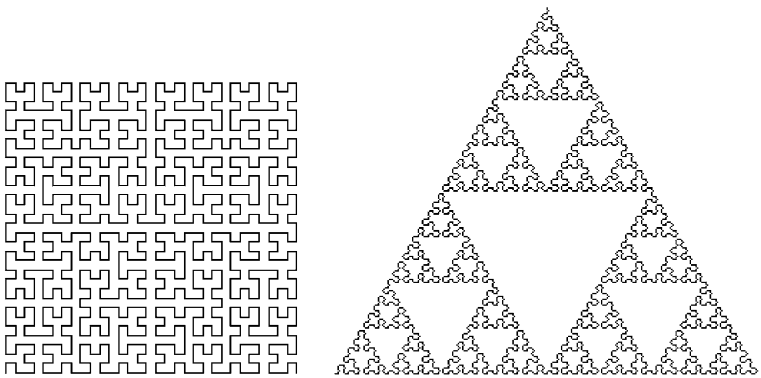


Fig. 8.5 Hilbert curve (5th generation) and Sierpinski arrowhead (7th generation)

It is remarkable that no Dragon curve intersects itself. There seems to be one such intersection in the 4th generation Dragon curve of Fig. 8.3 and many more in those of higher generations. However, that is not really the case, as demonstrated by using rounded instead of sharp corners. Figure 8.4 illustrates this for an 8th generation Dragon curve.

It is interesting to note that all the curves we have seen so far share one common characteristic, that is, each curve reflects the exact trace of the turtle's movement and is drawn essentially as one long and curved line. This is because the turtle always moves forward and draws by executing the F character command.

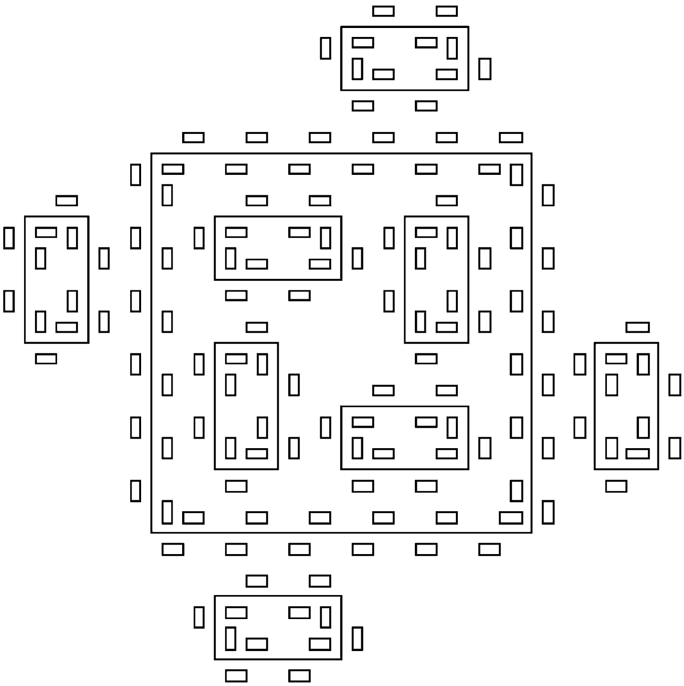


Fig. 8.6 Second generation Islands

Moving Without Drawing and f-Strings

Sometimes, it is desirable to keep some of the curve components at proper distances from each other. This implies that such a curve is not connected. We therefore need to define a forward moving action for the turtle without drawing:

- *f* - *move forward the distance D without drawing a line*.

The f-string, for which we already reserved a position (just after the F-string) indicates how each lower-case *f* is to be expanded. By using an f-string other than *nil*, we are able to generate an image with a combination of islands and lakes as shown in Fig. 8.6, based on the following grammar:

$$(F+F+F+F, F+f-FF+F+FF+Ff+FF-f+FF-F-FF-Ff-FFF,\\ ffffff, nil, nil, 90)$$

Note that here the parameters X-string and Y-string are unused and therefore written as *nil*.

To summarize, we have introduced the following six parameters into our string grammar:

1. The axiom from which the turtle starts;
2. The F-string for producing strings from *F* that instructs the turtle to move forward while drawing;
3. The f-string for producing strings from *f*, that instructs the turtle to move forward without drawing;
4. The X-string for producing strings from *X*, that does not affect the turtle;
5. The Y-string for producing strings from *Y*, that does not affect the turtle; and
6. The angle at which the turtle should turn.

In principle, more parameters may be introduced if the above six parameters cannot express new types of curves. On the other hand, a grammar does not have to use all the introduced parameters since, as we have seen, a *nil* can be used to represent an unused parameter.

Branching

With all the curves we have seen so far, one may observe the following phenomenon. The turtle is always moving forward along a curved line. It sometimes draws (seeing an *F* character) and sometimes does not draw (seeing an *f* character). The turtle never turns back to where it has visited before, since it cannot remember its previous positions. This implies that the turtle is unable to branch off from any of the previous curve positions and draw branching lines.

To make the turtle remember places that it has visited, we need to introduce the concept of the turtle's *state*. Let us call the turtle's current position together with its direction the *state* of the turtle. In other words, a state is defined by the values of a location in the drawing space and the angle at the location. To enhance the drawing power of string production rules, we allow the turtle to keep its states and return to any of them later by introducing two more character commands:

- [- *store the current state of the turtle*
-] - *restore the turtle's previously stored state*

These two characters, however, do not form strings of their own and thus do not require new production rules. They merely participate in the F-, f-, X-, and Y-strings to instruct how the turtle should behave.

The most appropriate data structure to implement the store and restore operations is a stack. Upon meeting a [character, the turtle pushes its current state onto the stack. When encountering a] character, the turtle pops its previous state from the stack and starts from the previous position and direction to continues its journey. Having empowered the turtle with the capability of returning and restarting from its previous states, we are able to draw curves with branches, such as trees defined in the following string grammars:

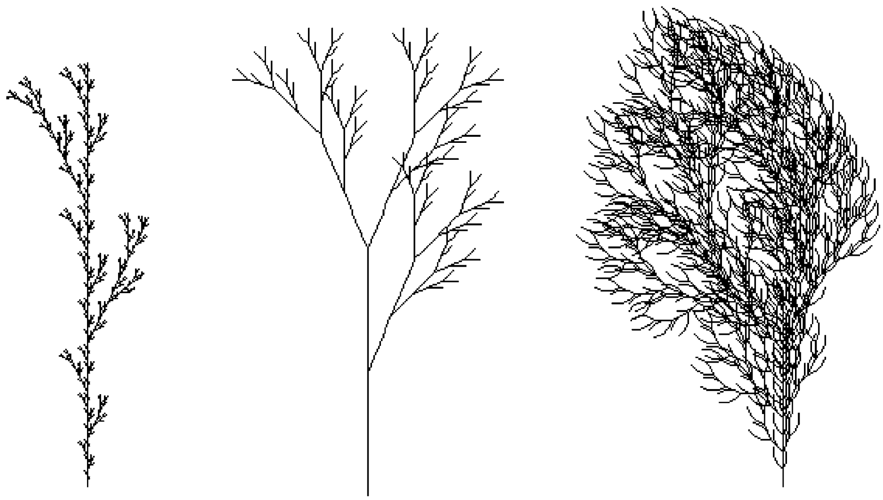


Fig. 8.7 Example of fractal trees: Tree1 (4th generation), Tree2 (5th-generation) and Tree3 (4th generation)

$$\text{Tree1} : (F, F[+F]F[-F]F, nil, nil, nil, 25.7)$$
$$\text{Tree2} : (X, FF, nil, F[+X]F[-X] + X, nil, 20.0)$$
$$\text{Tree3} : (F, FF - [-F + F + F] + [+F - F - F], nil, nil, nil, 22.5)$$

These grammars were used to obtain the trees shown in Fig. 8.7.

If the line thickness of each branch is set in proportion of its distance to the root, and also a small fraction of randomness is applied to the turning angle, more realistic looking trees would be produced.

Most figures of this section were produced by the program below, with input files to be discussed after this program:

```
// FractalGrammars.java
import java.awt.*;
import java.awt.event.*;

public class FractalGrammars extends Frame {
   public static void main(String[] args) {
      if (args.length == 0)
         System.out.println("Use filename as program argument.");
      else
         new FractalGrammars(args[0]);
   }
```

```
   FractalGrammars(String fileName) {
      super("Click left or right mouse button to change the level");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(800, 600);
      add("Center", new CvFractalGrammars(fileName));
      setVisible(true);
   }
}

class CvFractalGrammars extends Canvas {
   String fileName, axiom, strF, strf, strX, strY;
   int maxX, maxY, level = 1;
   double xLast, yLast, dir, rotation, dirStart, fxStart, fyStart,
         lengthFract, reductFact;

   void error(String str) {
      System.out.println(str);
      System.exit(1);
   }

   CvFractalGrammars(String fileName) {
      Input inp = new Input(fileName);
      if (inp.fails())
         error("Cannot open input file.");
      axiom = inp.readString(); inp.skipRest();
      strF = inp.readString(); inp.skipRest();
      strf = inp.readString(); inp.skipRest();
      strX = inp.readString(); inp.skipRest();
      strY = inp.readString(); inp.skipRest();
      rotation = inp.readFloat(); inp.skipRest();
      dirStart = inp.readFloat(); inp.skipRest();
      fxStart = inp.readFloat(); inp.skipRest();
      fyStart = inp.readFloat(); inp.skipRest();
      lengthFract = inp.readFloat(); inp.skipRest();
      reductFact = inp.readFloat();
      if (inp.fails()) error("Input file incorrect.");

      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent evt) {
            if ((evt.getModifiers() & InputEvent.BUTTON3_MASK) != 0) {
               level--; // Right mouse button decreases level
               if (level < 1) level = 1;
            } else
```

```
          level++; // Left mouse button increases level
        repaint();
      }
    });

  }

  Graphics g;

  int iX(double x) {return (int) Math.round(x);}
  int iY(double y) {return (int) Math.round(maxY - y);}

  void drawTo(Graphics g, double x, double y) {
     g.drawLine(iX(xLast), iY(yLast), iX(x), iY(y));
     xLast = x; yLast = y;
  }

  void moveTo(Graphics g, double x, double y) {
     xLast = x; yLast = y;
  }

  public void paint(Graphics g) {
     Dimension d = getSize();
     maxX = d.width - 1; maxY = d.height - 1;
     xLast = fxStart * maxX; yLast = fyStart * maxY;
     dir = dirStart; // Initial direction in degrees
     turtleGraphics(g, axiom, level, lengthFract * maxY);
  }

  public void turtleGraphics(Graphics g, String instruction,
        int depth, double len) {
     double xMark = 0, yMark = 0, dirMark = 0;
     for (int i = 0; i < instruction.length(); i++) {
        char ch = instruction.charAt(i);
        switch (ch) {
        case 'F': // Step forward and draw
        // Start: (xLast, yLast), direction: dir, steplength: len
        if (depth == 0) {
          double rad = Math.PI / 180 * dir, // Degrees -> radians
          dx = len * Math.cos(rad), dy = len * Math.sin(rad);
          drawTo(g, xLast + dx, yLast + dy);
        } else
          turtleGraphics(g, strF, depth - 1, reductFact * len);
        break;
```

```
        case 'f': // Step forward without drawing
          // Start: (xLast, yLast), direction: dir, steplength: len
          if (depth == 0) {
            double rad = Math.PI / 180 * dir, // Degrees -> radians
            dx = len * Math.cos(rad), dy = len * Math.sin(rad);
            moveTo(g, xLast + dx, yLast + dy);
          } else
            turtleGraphics(g, strf, depth - 1, reductFact * len);
          break;
        case 'X':
          if (depth > 0)
            turtleGraphics(g, strX, depth - 1, reductFact * len);
          break;
        case 'Y':
          if (depth > 0)
            turtleGraphics(g, strY, depth - 1, reductFact * len);
          break;
        case '+': // Turn right
          dir -= rotation;
          break;
        case '-': // Turn left
          dir += rotation;
          break;
        case '[': // Save position and direction
          xMark = xLast; yMark = yLast;
          dirMark = dir;
          break;
        case ']': // Back to saved position and direction
          xLast = xMark; yLast = yMark;
          dir = dirMark;
          break;
        }
      }
    }
}
```

The most essential input data consist of the grammar, for example,

$$(X, F, nil, X+YF+, -FX-Y, 90)$$

for the Dragon curve. In addition, the following five values would help in obtaining desirable results:

- *The direction in which the turtle starts, specified as the angle, in degrees, relative to the positive x-axis.*

- *The distance between the left window boundary and the start point, expressed as a fraction of the window width.*
- *The distance between the lower window boundary and the start point, expressed as a fraction of the window height.*
- *The length of a single line segment in the first generation, expressed as a fraction of the window height.*
- *A factor to reduce the length in each next generation, to prevent the image from growing outside the window boundaries.*

Setting the last two values can be regarded as tuning, so they were found experimentally. We supply all these data, that is, five strings and six real numbers, in a file, the name of which is supplied as a program argument. For example, to produce the Dragon curve, we can enter this command to start the program:

```
java FractalGrammars Dragon.txt
```

where the file *Dragon.txt* is listed below, showing the grammar $(X, F, X+YF+, -FX-Y, 90)$ followed by the five values just discussed:

```
"X"       // Axiom
"F"       // strF
""        // strf
"X+YF+"   // strX
"-FX-Y"   // strY
90        // Angle of rotation
0         // Initial direction of turtle (east)
0.5       // Start at x = 0.5 * width
0.5       // Start at y = 0.5 * height
0.6       // Initial line length is 0.6 * height
0.6       // Reduction factor for next generation
```

As you can see, strings are supplied between a pair of quotation marks, and comment is allowed at the end of each line. Instead of *nil*, we write the empty string "". Although the initial direction of the turtle is specified as east, the first line is drawn in the direction south. This is because of the axiom "X", which causes initially $strX =$ "$X+YF+$" to be used, where X and Y are ignored. So actually "$+F+$" is executed by the turtle. As we know, the initial + causes the turtle to turn right before the first line is drawn due to F, so this line is drawn downward instead of from left to right.

The other curves were produced in a similar way. Both the grammars and the five additional parameter values, as discussed above, for these curves are listed in the following table:

Dragon	$(X, F, nil, X+YF+, -FX-Y, 90)$	0	0.5	0.5	0.6	0.6
Hilbert	$(X, F, nil, -YF+XFX+FY-, +XF-YFY-FX+, 90)$	0	0.25	0.25	0.8	0.47
Sier-pinski	$(YF, F, nil, YF+XF+Y, XF-YF-X, 60)$.	0	0.33	0.5	0.38	0.51
Islands	$(F+F+F+F, F+f-FF+F+FF+Ff+FF-f$ $+FF-F-FF-Ff-FFF, ffffff, nil, nil, 90)$	0	0.25	0.65	0.2	0.2
Tree 1	$(F, F[+F]F[-F]F, nil, nil, nil, 25.7)$	90	0.5	0.05	0.7	0.34
Tree 2	$(X, FF, nil, F[+X]F[-X]+X, nil, 20.0)$	90	0.5	0.05	0.45	0.5
Tree 3	$(F, FF-[-F+F+F]+[+F-F-F], nil, nil,$ $nil, 22.5)$	90	0.5	0.05	0.25	0.5

When discussing the Tree examples with branches, we suggested that a stack would be used to push and pop states, while there seem to be no stack structure in the program. However, there is a local variables *xMark* in the recursive method *turtleGraphics*, and for each recursive call a version of this variable is stored on a system stack. In other words, the use of a stack is implicit in this program.

Further Extension

The above template and program provide a general framework for drawing grammar-based fractal images. You could easily extend the above program to include more strings to draw more complex and interesting curves. One example is a curve called *Penrose tiling*, originally proposed by Roger Penrose in 1974 [21]. To draw a Penrose curve, we could simply add two more strings, say U-string and V-string, introducing and using the variables *strU* and *strV* in the program in exactly the same way as *strX* and *strY*. We could then specify the curve in a file *Penrose.txt* as the following:

```
"[Y]++[Y]++[Y]++[Y]++[Y]"     // Axiom
""                            // strF
""                            // strf
"UF++VF----YF[-UF----XF]++"   // strX
"+UF--VF[---XF--YF]+"         // strY
"-XF++YF[+++UF++VF]-"         // strU
"--UF++++XF[+VF++++YF]--YF"   // strV
36                            // Angle of rotation
18                            // Initial direction of turtle
0.5                           // Start at x = 0.5 * width
0.5                           // Start at y = 0.5 * height
0.5                           // Initial factor for length
0.6                           // Reduction factor
```

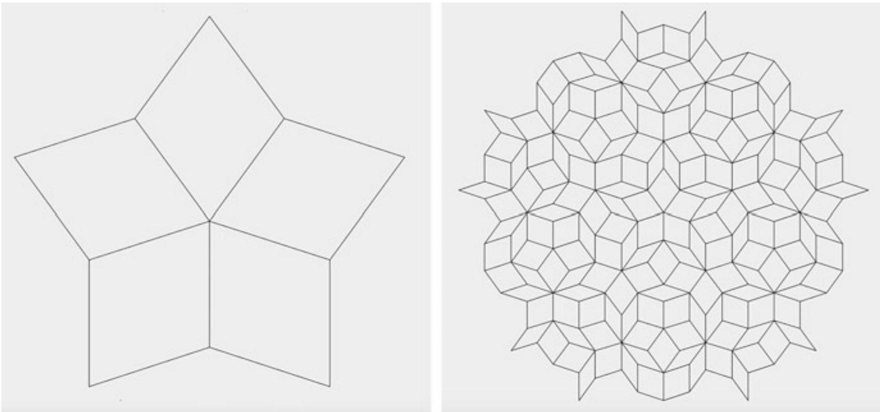


Fig. 8.8 Penrose tiling (*left*: 1st generation; *right*: 4th generation)

By inserting string processing codes for both U- and V-strings into the above *turtleGraphics* method, re-compile the entire *FractalGrammars.java* program and run.

```
java FractalGrammars Penrose.txt
```

we would generate the Penrose curves as shown in Fig. 8.8. For other curves, more angles may be added as additional parameters in the template in a similar fashion.

8.3 Mandelbrot Set

The *Mandelbrot set*, named after Polish-born French mathematician Benoît Mandelbrot, is a fractal. Recall that a fractal curve reveals small-scale details similar to the large-scale characteristics. Although the Mandelbrot set is self-similar at different scales, the small scale details are not *identical* to the whole. Also, the Mandelbrot set is infinitely complex. Yet the process of generating it is based on an extremely simple equation involving complex numbers. Figure 8.9 shows a view of the Mandelbrot set on the left. The outline is a fractal curve that can be zoomed in forever on any part for a close-up view, as the right part of Fig. 8.9 illustrates. Even parts of the image that appear quite smooth show a jagged outline consisting of many tiny copies of the Mandelbrot set. For example, as displayed in Fig. 8.9, there is a large and black region like a cardioid near the center with a circle joining its left-hand side. The region that the cardioid and the circle are joined together appears smooth. When the region is magnified, however, the detailed structure becomes apparent and shows many fascinating details that were not visible in the original picture. In theory, the zooming can be repeated forever, since the border is "infinitely complex".

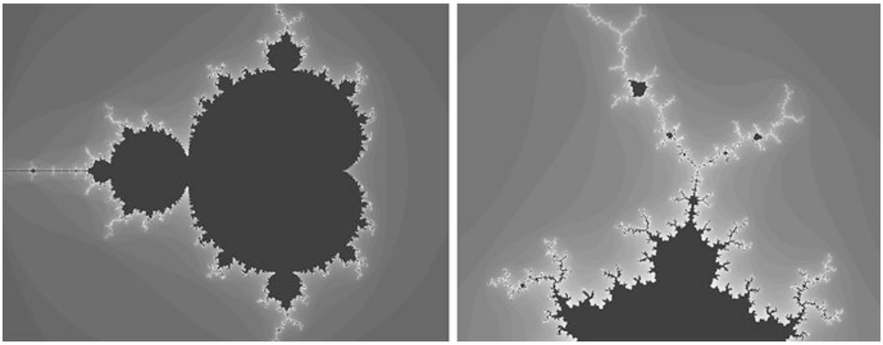


Fig. 8.9 Mandelbrot set and magnified detail

The Mandelbrot set is a set M of complex numbers defined in the following way:

$$M = \left\{ c \in C | \lim_{n \to \infty} z_n \neq \infty \right\}$$

where C is the set of all complex numbers and, for some constant c, the sequence z_0, z_1, ... is defined as follows:

$$\begin{aligned} z_0 &= 0 \\ z_{n+1} &= {z_n}^2 + c \end{aligned}$$

That is, the Mandelbrot set is the set of all complex numbers c which fulfill the condition described above. In other words, if the sequence z_0, z_1, z_2, ... does not approach infinity, then c belongs to the set. Given a value c, the system generates a sequence of values called the *orbit* of the start value 0:

$$\begin{aligned} z_0 &= 0 \\ z_1 &= {z_0}^2 + c = c \\ z_2 &= {z_1}^2 + c = c^2 + c \\ z_3 &= {z_2}^2 + c = \left(c^2 + c\right)^2 + c \\ z_4 &= {z_3}^2 + c = \left(\left(c^2 + c\right)^2 + c\right)^2 + c \\ &\ldots \end{aligned}$$

As soon as an element of the sequence $\{z_n\}$ is at a distance greater than 2 from the origin, it is certain that the sequence tends to infinity. A proof of this goes beyond the scope of this book. If the sequence does not approach infinity and therefore remains at a distance of at most 2 from the origin forever, then the point c is in the Mandelbrot set. If any z_n is farther than 2 from the origin, then the point c is not in the set.

Now let us consider how to generate a Mandelbrot image. Since the Mandelbrot set is a set of complex numbers, first we have to find these numbers that are part of the set. To do this we need a test that will determine if a given number is inside the set or outside. The test is applied to complex numbers z_n computed as $z_{n+1} = z_n^{\ 2} + c$. The constant c does not change during the testing process. As the number being tested, c is the point on the complex plane that will be plotted when the testing is complete. This plotting will be done in a color that depends on the test result. For some value n_{Max}, say 30, we start computing z_1, z_2, ... until either we have computed z_n for $n = n_{Max}$, or we have found a point $z_n (n \leq n_{Max})$ whose distance from the origin O is greater than 2. In the former case, having computed n_{Max} elements of the sequence, none of which is farther than a distance 2 away from O, we give up, consider the point c belonging to the Mandelbrot set, and plot it in black. In the latter case, the point z_n going beyond the distance 2, we plot the point c in a color that depends on the value of n.

Let us now briefly discuss complex arithmetic, as far as we need it here, for those who are unfamiliar with this subject. Complex numbers are two-dimensional by nature. We may regard the complex number

$$z = x + yi$$

as the real number pair (x, y). It is customary to refer to x as the *real part* and to y as the *imaginary part* of z. We display z in the usual way, with x drawn along the horizontal axis and y along the vertical axis. Addition of two complex numbers is the same as that for vectors:

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$$

By contrast, multiplication of complex numbers is rather complicated:

$$(x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$$

It follows that for

$$z = x + yi$$

we have

$$z^2 = \left(x^2 - y^2\right) + 2xyi$$

Although we do not need this for our purpose, we may as well note that by setting $x = 0$ and $y = 1$, giving $0 + 1 \times i = i$, we find

$$i^2 = \left(0^2 - 1^2\right) + 2 \times 0 \times 1 \times i = -1$$

which explains that the symbol i is often referred to as the square root of -1:

$$i = \sqrt{} - 1$$

The distance of the complex number $z = x + yi$ from the origin O is called the *absolute value* or *modulus* of z and is denoted as $|z|$. It follows that

$$| z |= \sqrt{x^2 + y^2}$$

Remember, $|z|^2 = x^2 + y^2$, while $z^2 = (x^2 - y^2) + 2xyi$ so, in general, $|z|^2$ is unequal to z^2. This very brief introduction to complex numbers cannot replace a thorough treatment as found in mathematics textbooks, but it will be sufficient for our Mandelbrot subject.

In the algorithm for Mandelbrot fractals, when computing each successive value of z, we want to know if its distance from the origin exceeds 2. To calculate this distance, usually denoted as $|z|$, we add the square of its distance from the x-axis (the horizontal real axis) to the square of its distance from the y-axis (the vertical imaginary axis) and then take the square root of the result. The computation of the square root operation can be saved by just checking whether the sum of these squares is greater than 4. In other words, for

$$z = x + yi$$

we perform the test

$$|z|^2 > 4$$

which is expressed in terms of real numbers as

$$x^2 + y^2 > 4$$

Now to compute each new value z using $z_{n+1} = {z_n}^2 + c$, let us write $Re(c)$ and $Im(c)$ for the real and imaginary parts of c. It then follows from

$$z^2 = (x + yi)^2 = \left(x^2 - y^2\right) + (2xy)i$$

that the real and imaginary parts of each element z_{n+1} are found as follows:

$$\begin{aligned} &\text{real part} &&: x_{n+1} = {x_n}^2 - {y_n}^2 + \mathrm{Re}(c) \\ &\text{imaginary part} &&: y_{n+1} = 2x_n y_n + \mathrm{Im}(c) \end{aligned}$$

As we increment n, the value of $|z_n|^2$ will either stay equal to or below 4 forever, or eventually surpass 4. Once $|z_n|^2$ surpasses 4, it will increase forever. In the former case, where the $|z_n|^2$ stays small, the number c being tested is part of the Mandelbrot set. In the latter case, when $|z_n|^2$ eventually surpasses 4, the number c is not part of the Mandelbrot set.

Implementation in Java

To display the whole Mandelbrot set image properly, we need some mapping

$$x_{\text{Pix}} \rightarrow x$$
$$y_{\text{Pix}} \rightarrow y$$

to convert the device coordinates $(x_{\text{Pix}}, y_{\text{Pix}})$ to the real and imaginary parts x and y of the complex number $c = x + yi$. In the program, the variables x_{Pix} and y_{Pix} are of type *int*, while x and y are of type *double*. We will use the following ranges for the device coordinates:

$$0 \leq x_{\text{Pix}} < \text{w}$$
$$0 \leq y_{\text{Pix}} < \text{h}$$

where we obtain the width w and height h of the drawing rectangle in the usual way:

```
w = getSize().width;
h = getSize().height;
```

For x and y we have

$$minRe \leq x \leq maxRe$$
$$minIm \leq y \leq maxIm$$

The user will be able to change these boundary variables *minRe*, *maxRe*, *minIm* and *maxIm* by dragging the left mouse button. Their default values are $minRe0 = -2$, $maxRe0 = +1$, $minIm0 = -1$, $maxIm0 = +1$, which will at any time be restored when the user presses the right mouse button. Using the variable *factor*, computed in Java as

```
factor = Math.max((maxRe - minRe)/w, (maxIm - minIm)/h);
```

we can perform the above mapping from $(x_{\text{Pix}}, y_{\text{Pix}})$ to (x, y) by computing

$$x = minRe + factor \times x_{\text{Pix}}$$
$$y = minIm + factor \times y_{\text{Pix}}$$

For every device coordinate pair $(x_{\text{Pix}}, y_{\text{Pix}})$ of the window, the associated point $c = x + \text{i}y$ in the complex plane is computed in this way. Then, in up to n_{Max} iterations, we determine whether this point belongs to the Mandelbrot set. If it does, we display the original pixel (x_{Pix}, y_{Pix}) in black; otherwise, we plot this pixel in a shade of red that depends on n, the number of iterations required to decide that the point is outside the Mandelbrot set. The expression $100 + 155^{*}n/n_{Max}$ ensures that this color value will not exceed its maximum value 255. If n_{Max} is set larger than that in the program, program execution will take more time but the quality of the image would improve. The implementation of this can by found in the *paint* method at the bottom of the following program, after which we will discuss the implementation of cropping and zooming.

```
// MandelbrotZoom.java: Mandelbrot set, cropping and zooming in.
import java.awt.*;
import java.awt.event.*;

public class MandelbrotZoom extends Frame {
   public static void main(String[] args) {new MandelbrotZoom();}

   MandelbrotZoom() {
      super("Drag left mouse button to crop and zoom. " +
            "Click right mouse button to restore.");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      setSize(800, 600);
      add("Center", new CvMandelbrotZoom());
      setVisible(true);
   }
}

class CvMandelbrotZoom extends Canvas {
   final double minRe0 = -2.0, maxRe0 = 1.0,
                minIm0 = -1.0, maxIm0 = 1.0;
   double minRe = minRe0, maxRe = maxRe0,
          minIm = minIm0, maxIm = maxIm0, factor, r;
   int n, xs, ys, xe, ye, w, h;

   void drawWhiteRectangle(Graphics g) {
      g.drawRect(Math.min(xs, xe), Math.min(ys, ye),
                 Math.abs(xe - xs), Math.abs(ye - ys));
   }

   boolean isLeftMouseButton(MouseEvent e) {
      return (e.getModifiers() & InputEvent.BUTTON3_MASK) == 0;
   }

   CvMandelbrotZoom() {
      addMouseListener(new MouseAdapter() {
         public void mousePressed(MouseEvent e) {
            if (isLeftMouseButton(e)) {
               xs = xe = e.getX(); // Left button
```

```
               ys = ye = e.getY();
            } else {
               minRe = minRe0; // Right button
               maxRe = maxRe0;
               minIm = minIm0;
               maxIm = maxIm0;
               repaint();
            }
         }

         public void mouseReleased(MouseEvent e) {
            if (isLeftMouseButton(e)) {
               xe = e.getX(); // Left mouse button released
               ye = e.getY(); // Test if points are really distinct:
               if (xe != xs && ye != ys) {
               int xS = Math.min(xs, xe), xE = Math.max(xs, xe),
                       yS = Math.min(ys, ye), yE = Math.max(ys, ye),
                       w1 = xE - xS, h1 = yE - yS, a = w1 * h1,
                       h2 = (int) Math.sqrt(a / r), w2 = (int) (r * h2),
                       dx = (w2 - w1) / 2, dy = (h2 - h1) / 2;
               xS -= dx; xE += dx;
               yS -= dy; yE += dy; // aspect ration corrected
               maxRe = minRe + factor * xE;
               maxIm = minIm + factor * yE;
               minRe += factor * xS;
               minIm += factor * yS;
               repaint();
               }
            }
         }
      });

      addMouseMotionListener(new MouseMotionAdapter() {
         public void mouseDragged(MouseEvent e) {
            if (isLeftMouseButton(e)) {
               Graphics g = getGraphics();
               g.setXORMode(Color.black);
               g.setColor(Color.white);
               if (xe != xs || ye != ys)
                  drawWhiteRectangle(g); // Remove old rectangle:
               xe = e.getX(); ye = e.getY();
               drawWhiteRectangle(g); // Draw new rectangle:
            }
         }
      });
   }
```

```
   public void paint(Graphics g) {
      w = getSize().width; h = getSize().height;
      r = w / h; // Aspect ratio, used in mouseReleased
      factor = Math.max((maxRe - minRe) / w, (maxIm - minIm) / h);
      for (int yPix = 0; yPix < h; ++yPix) {
         double cIm = minIm + yPix * factor;
         for (int xPix = 0; xPix < w; ++xPix) {
            double cRe = minRe + xPix * factor, x = cRe, y = cIm;
            int nMax = 100, n;
            for (n = 0; n < nMax; ++n) {
               double x2 = x * x, y2 = y * y;
               if (x2 + y2 > 4) break; // Outside
               y = 2 * x * y + cIm;
               x = x2 - y2 + cRe;
            }
            g.setColor(n == nMax ? Color.black            // Inside
               : new Color(100 + 155 * n / nMax, 0, 0)); // Outside
            g.drawLine(xPix, yPix, xPix, yPix);
         }
      }
   }
}
```

As indicated in the title bar (implemented by a call to *super* at the beginning of the *MandelbrotZoom* constructor), the user can zoom in by using the left mouse button. By dragging the mouse with the left button pressed down, a rectangle appears, as shown in Fig. 8.10, with one of its corners at the point first clicked and the opposite one denoting the current mouse position. This process is sometimes referred to as *rubber banding*.

When the user releases the mouse button, the picture is cropped so that everything outside the rectangle is removed and the contents inside are enlarged and displayed in the full window. This cropping is therefore combined with zooming in, as Fig. 8.11 illustrates. The user can continue zooming in by cropping in the same way forever. It would be awkward if there were no way of either zooming out or returning to the original image. In program *MandelbrotZoom.java* the latter is possible by clicking the right mouse button.

To implement this cropping rectangle, we use two opposite corner points of it. The start point (*xs*, *ys*) is where the user clicks the mouse to start dragging, while the opposite corner is the endpoint (*xe*, *ye*). There are three Java methods involved:

mousePressed: to define both the start point (*xs*, *ys*) and initial position of the endpoint (*xe*, *ye*);

mouseDragged: to update the endpoint (*xe*, *ye*), removing the old rectangle and drawing the new one;

mouseReleased: to compute the logical boundary values *minRe*, *maxRe*, *minIm* and *maxIm* on the basis of (*xs*, *ys*) and (*xe*, *ye*).

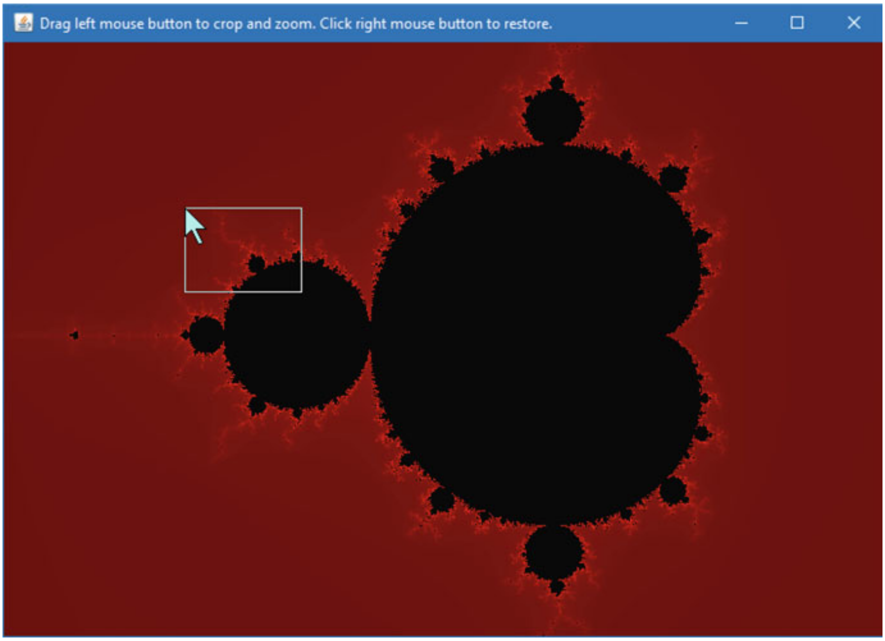


Fig. 8.10 Cropping and zooming in

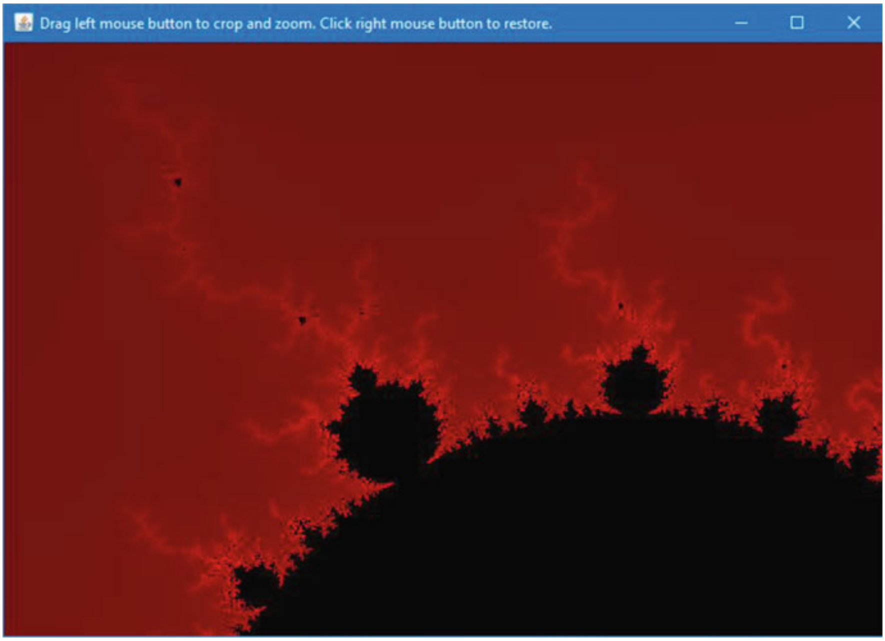


Fig. 8.11 Result of cropping and zooming in

In *mouseDragged*, drawing and removing the cropping rectangle is done using the XOR mode which was briefly discussed at the end of Sect. 4.3. Here we use two calls

```
g.setXORMode(Color.black);
g.setColor(Color.white);
```

after which we draw both the old and the new rectangles. In *mouseReleased*, some actions applied to the cropping rectangle require explanation. As we want the zooming to be isotropic, we have to pay attention to the cropping rectangle having an aspect ratio

$$r_1 = w_1 : h_1$$

different from the aspect ratio

$$r = w : h$$

of the (large) drawing rectangle. For example, the cropping rectangle may be in 'portrait' format (with $w_1 < h_1$), while the drawing rectangle has the 'landscape' characteristic (with $w > h$). The simplest way to deal with this case would be to cut off a portion of the cropping rectangle at the top or the bottom. However, that would lead to a result that may be unexpected and undesirable for the user. We will therefore replace the cropping rectangle with one that has the same center and the same area $a = w_1 h_1$, but an aspect ratio of r (mentioned above) instead of r_1. The dimensions of this new rectangle will be $w_2 \times h_2$ instead of $w_1 \times h_1$. To find w_2 and h_2 we solve

$$w_2 h_2 = a \quad (= \text{area of cropping rectangle defined by the user})$$
$$w_2 : h_2 = r \quad (= \text{aspect ratio}\, w/h\, \text{of drawing rectangle})$$

giving

$$h_2 = \sqrt{\frac{a}{r}}$$
$$w_2 = rh_2$$

Then we add the correction term $\Delta x = ½(w_2 - w_1)$ to the x-coordinate xE of the right cropping-rectangle edge and subtract it from the x-coordinate xS of the corresponding left edge. After performing similar operations on yS and yE, the resulting new rectangle with top-left corner point (xS, yS) and bottom-right corner point (xE, yE) is about as large as user-defined rectangle but in shape similar to the large drawing rectangle. We then have to compute the corresponding coordinates in the complex plane, using the mapping discussed a short while ago. Since the new logical right boundary values *maxRe* and *maxIm* should correspond to *xE* and *yE*, we compute

```
maxRe = minRe + factor * xE;
maxIm = minIm + factor * yE;
```

Similarly, to obtain the new left boundary values, we have to add *factor* × *xS* and *factor* × *yS* to the old values *minRe* and *minIm*, respectively, giving the slightly more cryptic statements

```
minRe += factor * xS;
minIm += factor * yS;
```

8.4 Julia Set

Associated with every point in the complex plane is a set somewhat similar to the Mandelbrot set called a *Julia set*, named after the French mathematician Gaston Julia. To produce an image of a Julia set, we use the same iteration $z_{n+1} = z_n^{\ 2} + c$, this time with a constant value of c but with a starting value z_0 derived from the coordinates (x_{Pix}, y_{Pix}) of the pixel displayed on the screen. We obtain interesting Julia sets if we choose points near the boundaries of the Mandelbrot set as starting values z_0. For example, we obtained Fig. 8.12 by taking $c = -0.76 + 0.084i$, which in the Mandelbrot set (Fig. 8.9) is the point near the top of the circle on the left.

Since the Mandelbrot set can be used to select c for the Julia set, it is said to form an index into the Julia set. Such an interesting relationship is also evidenced by the following fact. A Julia set is either connected or disconnected. For values of z_0 chosen from within the Mandelbrot set, we obtain connected Julia sets. That is, all

Fig. 8.12 Julia set, obtained by using $c = -0.76 - 0.084i$

the black regions are connected. Conversely, those values of z_0 outside the Mandelbrot set give disconnected Julia sets. The disconnected sets are often called *dust*, consisting of individual points no matter what resolution they are viewed at. If, in the program *MandelbrotZoom.java*, we replace the paint method with the one below (preferably also replacing the name *Mandelbrot* with *Julia* throughout the program), and specify the dimensions $w = 900$ and $h = 500$ in the call to *setSize*, we obtain a program which initially produces Fig. 8.12. Again, we can display all kinds of fascinating details by cropping and zooming. Note, in this program, c is a constant and the starting value z of the sequence is derived from the device coordinates (x_{Pix}, y_{Pix}).

```
   public void paint(Graphics g) {
      Dimension d = getSize();
      w = getSize().width;
      h = getSize().height;
      r = w/h;
      double cRe = -0.76, cIm = 0.084;
      factor = Math.max((maxRe - minRe)/w, (maxIm - minIm)/h);
      for(int yPix=0; yPix<h; ++yPix) {
         for(int xPix=0; xPix<w; ++xPix) {
            double x = minRe + xPix * factor,
                   y = minIm + yPix * factor;
            int nMax = 100, n;
            for (n=0; n<nMax; ++n) {
               double x2 = x * x, y2 = y * y;
               if (x2 + y2 > 4)
                  break;   // Outside
               y = 2 * x * y + cIm;
               x = x2 - y2 + cRe;
            }
           g.setColor(n == nMax ? Color.black            // Inside
               : new Color(100 + 155 * n / nMax, 0, 0)); // Outside
            g.drawLine(xPix, yPix, xPix, yPix);
         }
      }
   }
}
```

Exercises

8.1. Write a program to produce Dragon curves with rounded corners, as shown in Fig. 8.4.

8.2. Write a program to connect Koch curves, as shown in Fig. 8.2.

Fig. 8.13 A fractal tree with different branch thicknesses

8.3. To make bush curves (trees) appear more natural and pleasing, add randomness to the angles and lengths of the lines, and also draw lines with thickness. For example, the lower part of a tree is thicker than its upper part, as shown in Fig. 8.13.

8.4. Further to Exercise 8.3, add green leaves to the branches and make the leaves oriented toward the same directions as the branches.

8.5. Apply the method of cropping and zooming used in *MandelbrotZoom.java* to some other (non-trivial) graphics program of your own choice.

8.6. Write a program to draw the Mandelbrot set, and when clicked on the Mandelbrot image, a Julia set corresponding to the clicked point is drawn on a window aside of the Mandelbrot window. The clicked point in the form of $z = x + yi$ should also be displayed on the side window.

Appendix A: Interpolation of 1/*z*

In 3D computer graphics, given an object in 3D space (the real world), the usual task is to compute its image in 2D space (the screen). If a line segment PQ in 3D is given, we compute the images P′ and Q′ of its endpoints, and draw the line segment P′Q′. However, the reverse task, that is, computing the original point P in 3D space of a given image point P′ on the screen, is sometimes also required, as discussed both in Sect. 6.1 and Appendix C for the hidden-line algorithm and Sect. 6.4 for the Z-buffer algorithm.

In Fig. A.1 the viewer's eye is located in E, the origin of the *xz*-coordinate system, and the screen is at $z = -20$. For simplicity, we ignore the *y*-axis. The viewer sees four points with *z*-coordinates −40, −60, −80 and −100 on the left. Their image points on the screen have *x*-coordinates −25, −16.7, −12.5 and −10, respectively. Although the original points are equidistant (with distances 20), the images of far-away points are closer together than those of nearby points. The right half of this figure shows a different situation, which is about the main subject we want to discuss here. With a given line segment PQ, the images of the endpoints are computed from their positions $z = -100$ and $z = -40$, giving $x = 10$ and $x = 25$, respectively, for the image points P′ and Q′. Then P′Q′ is divided into three equal parts, so we have four equidistant image points. From the *x*-coordinates 15 and 20 of the two intermediate points, the *z*-coordinates −66.7 and −50 of the original points are then computed. The two intermediate image points are similar to (a great many and equidistant) pixels on an image line segment P′Q′ on a computer screen.

In this example, the relation between the *z*-coordinates of the original points and the *x*-coordinates of their images is very simple. This is because, on both the left and the right parts of Fig. A.1, the four original points lie on a line parallel to the *z*-axis, that is, perpendicular to the screen. For example, the two right triangles EFQ and EF′Q′ have the same angles so they are similar, and

$$\frac{\mathrm{QF}}{\mathrm{EF}} = \frac{\mathrm{Q'F'}}{\mathrm{EF'}}$$

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2

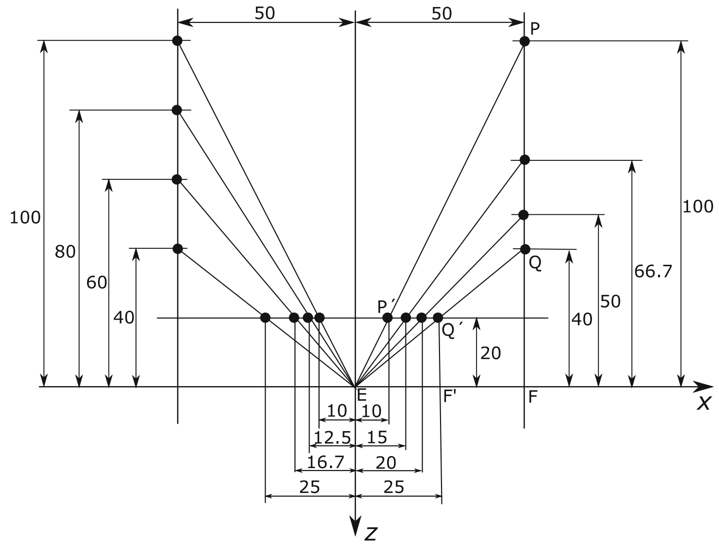
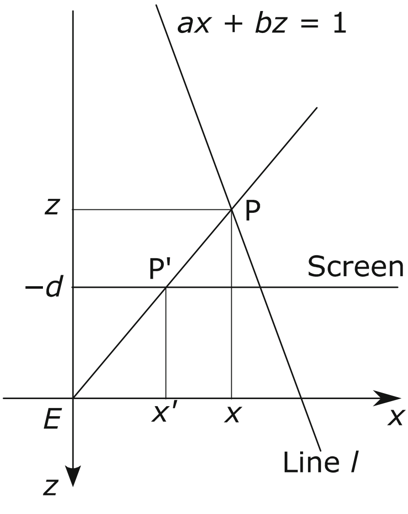


Fig. A.1 Points equidistant in 3D space on the *left* and equidistant in 2D space on the *right*

Fig. A.2 Central projection of point P on screen



so

$$x' \times (-z) = \mathrm{EF}' \times \mathrm{QF} = \mathrm{EF} \times \mathrm{Q}'\mathrm{F}' = 50 \times 20 = 1000$$

This also applies to the other pairs of similar triangles, so, if we know either x' of an image or z of an original point we can simply divide -1000 by it to find the other.

Unfortunately, things are less straightforward if the original lines are not parallel to the z-axis. In Fig. A.2 we have such a line l, with equation

$$ax + bz = 1 \tag{A.1}$$

The screen is represented by the line $z = -d$.

For a given projection P′ with x-coordinate x', our task is now to compute the z-coordinate of the original point P on line l. Geometrically, P is found as the intersection of l and an extension of EP′. Since the line through E and P′ has slope $-\,d/x'$ and passes through the origin E, we can easily write its equation

$$z = -\frac{d}{x'}x$$

where the minus sign is required because the positive z-axis points downward. We write this as

$$x = -\frac{z}{d}x'$$

and use this to substitute x in Eq. (A.1) of line l, obtaining

$$-a\frac{z}{d}x' + bz = 1$$

Exchanging both sides and dividing them by z gives

$$\frac{1}{z} = -\frac{a}{d}x' + b \tag{A.2}$$

which shows that $1/z$ is a linear function of x'. This is important because it enables us to perform *linear interpolation*. This term is used to indicate that, with a linear function

$$f(x) = mx + c$$

and three values x_A, x_B and x_I satisfying

$$x_I = x_A + \lambda(x_B - x_A)$$

for any value λ between 0 and 1, we can express $f(x_I)$ in $f(x_A)$ and $f(x_B)$ in the same way as x_I is expressed in x_A and x_B, that is,

$$f(x_I) = f(x_A) + \lambda\{f(x_B) - f(x_A)\}$$

For example, if x_I lies halfway between x_A and x_B, then $f(x_I)$ lies halfway between $f(x_A)$ and $f(x_B)$.

As we have seen in Eq. (A.2), $1/z$ is a linear function of x'. It then follows that we can find the z-coordinate of a point I on a line AB in 3D-space by using

$$\frac{1}{z_I} = \frac{1}{z_A} + \lambda\left(\frac{1}{z_B} - \frac{1}{z_A}\right) \tag{A.3}$$

where λ indicates the position of the projection I′ relative to A′ and B′:

$$x'_{\mathrm{I}} = x'_{\mathrm{A}} + \lambda\left(x'_{\mathrm{B}} - x'_{\mathrm{A}}\right)$$

Note that Eq. (A.3) does not contain any details of the line l. We have used this equation in Sect. 6.4 for the Z-buffer algorithm; it is also relevant to hidden-line removal, as discussed in Sect. 6.1 and Appendix C.

Let us now return to Fig. A.1 to apply linear interpolation to the second point from below on the right, with $z = -50$. We will denote this point as I and use A and B instead of P and Q to comply with Eq. (A.3). We will use no information other than $z_{\mathrm{A}} = -100$, $z_{\mathrm{B}} = -40$, $x_{\mathrm{A}}' = 10$, $x_{\mathrm{B}}' = 25$, and $x_{\mathrm{I}}' = 20$. This implies that we will use

$$\lambda = \frac{x'_{\mathrm{I}} - x'_{\mathrm{A}}}{x'_{\mathrm{B}} - x'_{\mathrm{A}}} = \frac{20 - 10}{25 - 10} = \frac{2}{3}$$

Then we apply Eq. (A.3), obtaining

$$\frac{1}{z_{\mathrm{I}}} = \frac{1}{-100} + \frac{2}{3}\left(\frac{1}{-40} - \frac{1}{-100}\right) = -\frac{1}{50}$$

so that $z_{\mathrm{I}} = -50$. Unlike our earlier discussion, we have now obtained this value without having used the equation of the line under discussion. Since in Fig. A.1, this line is parallel to the z-axis, the value of b in Eq. (A.1) is 0, so it has the equation $ax = 1$, where $a = 0.02$; in other words, $x = 50$. The situation of Fig. A.1 is a special case, with constant value $x'z$ ($= -1000$ for all four points on the right), enabling us to compute the z-values from x' in a trivial way. In the more general case, with nonzero values of b, we cannot use this, but, fortunately, linear interpolation is then available to compute these values efficiently.

A.1 A Different Notation

In the preceding discussion, we have written interpolation equations of the form

$$x'_{\mathrm{I}} = x'_{\mathrm{A}} + \lambda\left(x'_{\mathrm{B}} - x'_{\mathrm{A}}\right)$$

in which the variable x_{A}' occurs twice. We can instead use a different equation, in which each of the variables x_{A}' and x_{B}' occurs only once, at the price of using two occurrences of the parameter λ:

$$x'_{\mathrm{I}} = (1 - \lambda)x'_{\mathrm{A}} + \lambda x'_{\mathrm{B}}$$

Appendix B: Class *Obj3D*

The following is not a complete program. It is the code for a single class, *Obj3D*, which is discussed in Sect. 5.6 and used in the programs *Wireframe.java* of Sect. 5.7, *HLines.java* of Sect. 6.1, *Painter.java* of Sect. 6.3, and *ZBuf.java* of Sect. 6.4.

```
// Obj3D.java: A 3D object and its 2D representation.
// Uses: Point2D (Section 1.4), Point3D (Section 3.9),
//       Polygon3D, Input (Section 5.6).
import java.awt.*;
import java.util.*;

class Obj3D {
   Obj3D() {
      setSpecular(false);
   }

   private float rho, d, theta = 0.30F, phi = 1.3F, rhoMin, rhoMax,
         xMin, xMax, yMin, yMax, zMin, zMax, v11, v12, v13, v21, v22,
         v23, v32, v33, v43; // , xe, ye, ze, objSize;
   private Point2D imgCenter;
   private Vector<Point3D> w = new Vector<Point3D>();
                                 // World coordinates
   private Point3D[] e; // Eye coordinates
   private Point2D[] vScr; // Screen coordinates
   private Vector<Polygon3D> polyList = new Vector<Polygon3D>();
   private String fName = ""; // File name

   // Light vector, normalized, pointing to light source,
   // expressed in eye coordinates:
   private final double coordValue = 1 / Math.sqrt(3);
```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```
private double xL = -coordValue, yL = coordValue, zL = coordValue,
               zV = 1; // Vector V, eye coordinates

boolean read(String fName) {
   Input inp = new Input(fName);
   if (inp.fails())
      return failing();
   this.fName = fName;
   xMin = yMin = zMin = +1e30F;
   xMax = yMax = zMax = -1e30F;
   return readObject(inp); // Read from inp into obj
}

Vector<Polygon3D> getPolyList() {return polyList;}
String getFName() {return fName;}
Point3D[] getE() {return e;}
Point2D[] getVScr() {return vScr; }
Point2D getImgCenter() {return imgCenter;}
float getRho() {return rho;}
float getD() {return d;}

private boolean failing() {
   Toolkit.getDefaultToolkit().beep();
   return false;
}

private boolean readObject(Input inp) {
   for (;;) {
      int i = inp.readInt();
      if (inp.fails()) {
         inp.clear();
         break;
      }
      if (i < 0) {
         System.out.println(
               "Negative vertex number in first part of input file");
         return failing();
      }
      w.ensureCapacity(i + 1);
      float x = inp.readFloat(), y = inp.readFloat(),
           z = inp.readFloat();
      addVertex(i, x, y, z);
   }
   shiftToOrigin(); // Origin in center of object.
   char ch;
```

```
      int count = 0;
      do // Skip the line "Faces:"
      {
         ch = inp.readChar();
         count++;
      } while (!inp.eof() && ch != '\n');
      if (count < 6 || count > 8) {
         System.out.println("Invalid input file");
         return failing();
      }
      // Build polygon list:
      for (;;) {
         Vector<Integer> vnrs = new Vector<Integer>();
         for (;;) {
            int i = inp.readInt();
            if (inp.fails()) {
               inp.clear();
               break;
            }
            int absi = Math.abs(i);
            if (i == 0 || absi >= w.size() || w
                  .elementAt(absi) == null) {
               System.out.println(
                     "Invalid vertex number: " + absi +
                     " must be defined, nonzero and less than " + w.size());
               return failing();
            }
            vnrs.addElement(new Integer(i));
         }
         ch = inp.readChar();
         if (ch != '.' && ch != '#')
            break;
         // Ignore input lines with only one vertex number:
         if (vnrs.size() >= 2)
            polyList.addElement(new Polygon3D(vnrs));
      }
      inp.close();
      return true;
   }

   private void addVertex(int i, float x, float y, float z) {
      if (x < xMin) xMin = x;
      if (x > xMax) xMax = x;
      if (y < yMin) yMin = y;
```

```
      if (y > yMax) yMax = y;
      if (z < zMin) zMin = z;
      if (z > zMax) zMax = z;
      if (i >= w.size()) w.setSize(i + 1);
      w.setElementAt(new Point3D(x, y, z), i);
   }

   private void shiftToOrigin() {
      float xwC = 0.5F * (xMin + xMax), ywC = 0.5F * (yMin + yMax),
           zwC = 0.5F * (zMin + zMax);
      int n = w.size();
      for (int i = 1; i < n; i++)
         if (w.elementAt(i) != null) {
            w.elementAt(i).x -= xwC;
            w.elementAt(i).y -= ywC;
            w.elementAt(i).z -= zwC;
         }
      float dx = xMax - xMin, dy = yMax - yMin, dz = zMax - zMin;
      rhoMin = 0.6F * (float) Math.sqrt(dx * dx + dy * dy + dz * dz);
      rhoMax = 1000 * rhoMin;
      rho = 3 * rhoMin;
   }

   private void initPersp() {
      float costh = (float) Math.cos(theta),
            sinth = (float) Math.sin(theta),
            cosph = (float) Math.cos(phi),
            sinph = (float) Math.sin(phi);
      v11 = -sinth; v12 = -cosph * costh; v13 = sinph * costh;
      v21 = costh; v22 = -cosph * sinth; v23 = sinph * sinth;
      v32 = sinph; v33 = cosph;
      v43 = -rho;
   }

   float eyeAndScreen(Dimension dim) { // Called in Canvas class
     initPersp();
     int n = w.size();
     e = new Point3D[n];
     vScr = new Point2D[n];
     float xScrMin = 1e30F, xScrMax = -1e30F, yScrMin = 1e30F,
          yScrMax = -1e30F;
     for (int i = 1; i < n; i++) {
        Point3D P = w.elementAt(i);
```

```
      if (P == null) {
         e[i] = null;
         vScr[i] = null;
      } else {
        float x = v11 * P.x + v21 * P.y;
        float y = v12 * P.x + v22 * P.y + v32 * P.z;
        float z = v13 * P.x + v23 * P.y + v33 * P.z + v43;
        Point3D Pe = e[i] = new Point3D(x, y, z);
        float xScr = -Pe.x / Pe.z, yScr = -Pe.y / Pe.z;
        vScr[i] = new Point2D(xScr, yScr);
        if (xScr < xScrMin) xScrMin = xScr;
        if (xScr > xScrMax) xScrMax = xScr;
        if (yScr < yScrMin) yScrMin = yScr;
        if (yScr > yScrMax) yScrMax = yScr;
      }
    }
    float rangeX = xScrMax - xScrMin, rangeY = yScrMax - yScrMin;
    d = 0.95F * Math.min(dim.width / rangeX, dim.height / rangeY);
    imgCenter = new Point2D(d * (xScrMin + xScrMax) / 2,
        d * (yScrMin + yScrMax) / 2);
    for (int i = 1; i < n; i++) {
      if (vScr[i] != null) {
          vScr[i].x *= d;
          vScr[i].y *= d;
      }
    }
    return d * Math.max(rangeX, rangeY);
    // Maximum screen-coordinate range used in CvHLines for HP-GL
  }

  void planeCoeff() {
     int nFaces = polyList.size();

     for (int j = 0; j < nFaces; j++) {
       Polygon3D pol = polyList.elementAt(j);
       int[] nrs = pol.getNrs();
       if (nrs.length < 3) continue;
       int iA = Math.abs(nrs[0]), // Possibly negative
           iB = Math.abs(nrs[1]), // for HLines.
           iC = Math.abs(nrs[2]);
       Point3D A = e[iA], B = e[iB], C = e[iC];
       double u1 = B.x - A.x, u2 = B.y - A.y, u3 = B.z - A.z,
              v1 = C.x - A.x, v2 = C.y - A.y, v3 = C.z - A.z,
              a = u2 * v3 - u3 * v2, b = u3 * v1 - u1 * v3,
              c = u1 * v2 - u2 * v1,
              len = Math.sqrt(a * a + b * b + c * c), h;
```

```
            a /= len; b /= len; c /= len;
            h = a * A.x + b * A.y + c * A.z;
            pol.setAbch(a, b, c, h);
            if (u1 * v2 - u2 * v1 <= 0)
               continue; // backface
         }
      }

      boolean vp(Canvas cv, float dTheta, float dPhi, float fRho) {
         theta += dTheta;
         phi += dPhi;
         float rhoNew = fRho * rho;
         if (rhoNew >= rhoMin && rhoNew <= rhoMax)
            rho = rhoNew;
         else
            return false;
         cv.repaint();
         return true;
      }

      double kAmb, kDiff, kSpec;

      void setSpecular(Boolean isSpecular) {
         if (isSpecular) {
            kAmb = 0.2; kDiff = 0.7; kSpec = 0.2;
         }
         else {// Diffuse
            kAmb = 0.4; kDiff = 0.6; kSpec = 0.0;
         }
      }

      int colorCodePhong(double xN, double yN, double zN) {
         // Viewing vector V (from O to E, length 1):
         double colorAmbR = 1, colorAmbG = 1, colorAmbB = 0,
                colorDifR = 1, colorDifG = 1, colorDifB = 0,
                colorSpecR = 1, colorSpecG = 1, colorSpecB = 0;
         // Red (R) and green (G) without blue (B) gives yellow.

         // Ambient component:
         double illumAmbR = kAmb * colorAmbR,
                illumAmbG = kAmb * colorAmbG,
                illumAmbB = kAmb * colorAmbB;
```

```
    // Diffuse component:
    double inprodLN = Math.max(0, xL * xN + yL * yN + zL * zN),
           illumDiff = inprodLN * kDiff,
           illumDiffR = illumDiff * colorDifR,
           illumDiffG = illumDiff * colorDifG,
           illumDiffB = illumDiff * colorDifB;

    // Specular component:
    // Reflection vector R = 2(L . N)N - L
    // xR and yR would only be used to multiply them by xV and yV,
    // and these are zero since V points to the viewpoint E and we are
    // using eye coordinates, so computing xR and yR would be useless.
    double zR = 2 * inprodLN * zN - zL,
           dotProductVR = Math.max(0, zV * zR), // xV = yV = 0
           illumSpec = kSpec * Math.pow(dotProductVR, 16),
           illumSpecR = illumSpec * colorSpecR,
           illumSpecG = illumSpec * colorSpecG,
           illumSpecB = illumSpec * colorSpecB;

    // Sum of ambient, diffuse and specular illumination:
    double illumR = Math.min(1, illumAmbR + illumDiffR + illumSpecR),
           illumG = Math.min(1, illumAmbG + illumDiffG + illumSpecG),
           illumB = Math.min(1, illumAmbB + illumDiffB + illumSpecB);

    int red = (int) (255 * illumR),
        green = (int) (255 * illumG),
        blue = (int) (255 * illumB);
    return (red << 16) | (green << 8) | blue;
  }
}
```

Appendix C: Hidden-Line Tests and Implementation

This appendix presents the details of all nine tests for visibility used in the hidden-line removal algorithm in Sect. 6.1. It also introduces the output of the hidden-line removal algorithm in the HP-GL format and how this can be implemented in Java. The appendix finally lists Java source code *CvHLines.java* that implements the hidden-line removal algorithm.

C.1. Tests for Line Visibility

Test 1 (2D; Fig. C.1)

If neither P nor Q lies to the right of the leftmost one of the three vertices A, B and C (of triangle t) the triangle does not obscure PQ. This type of test is known as a *minimax* test: we compare the maximum of all x-coordinates of PQ with the minimum of all those of triangle t. Loosely speaking, we have now covered the case that PQ lies completely to the left of triangle t. In the same way, we deal with PQ lying completely to the right of t. Similar tests are performed for the minimum and maximum y-coordinates.

Test 2 (3D; Fig. C.2)

If PQ (in 3D-space) is identical with one of the edges of triangle t, this triangle does not obscure PQ. This test is done very efficiently by using the vertex numbers of P, Q, A, B and C. As we will see in a moment, in recursive calls, P or Q may be a computed point, which has no vertex number. It is therefore not always possible to perform this test.

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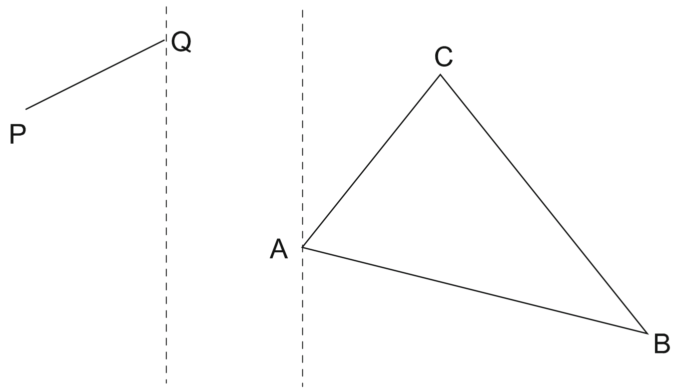
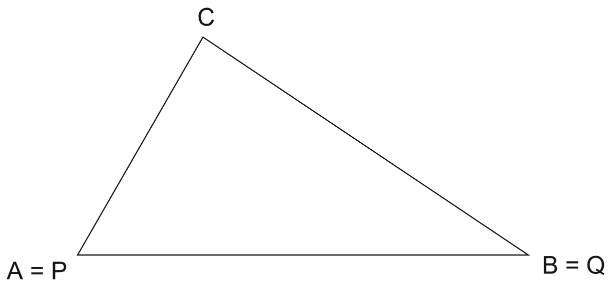


Fig. C.1 Test 1: both P and Q on the left of A, B and C

Fig. C.2 Test 2: PQ identical with AB



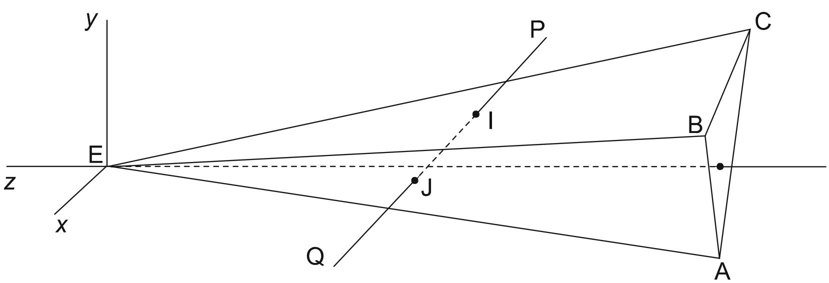


Fig. C.3 Test 3: z-coordinates of P and Q less than those of A, B and C

Test 3 (3D; Fig. C.3)

If neither P nor Q is farther away than the nearest of the three vertices A, B and C of triangle t, this triangle does not obscure PQ. This is a minimax test, like Test 1, but this time applied to the z-coordinates. Since the positive z-axis points to the left in Fig. C.3, the greater the z-coordinate of a point, the nearer this point is. Therefore, triangle ABC does not obscure PQ if the minimum of z_P and z_Q is greater than or equal to the maximum of z_A, z_B and z_C.

Test 4 (2D; Fig. C.4)

If, on the screen, the points P and Q lie on one side of the line AB while the third triangle vertex C lies on the other, triangle ABC does not obscure PQ. The lines BC and CA are dealt with similarly. This test is likely to succeed, but we perform it only after the previous three tests because it is rather expensive. Since the vertices A, B and C are counter-clockwise, the points P and C are on different sides of AB if and only if the point sequence ABP is clockwise. As we have seen in Sect. 2.3, this implies that we can use the static method *area*2 of class *Tools*2*D*. With *Point*2*D* objects *AScr*, *BScr*, *CScr* for the points A, B and C, the left-hand side in the comparison

```
Tools2D.area2(AScr, BScr, PScr) <= 0
```

is equal to twice the area of triangle ABP, preceded by a minus sign if (and only if) the sequence ABP is clockwise. This value is negative if P and C lie on different sides of AB, and it is zero if A, B and P lie on a straight line. If this comparison succeeds and the same applies to

```
Tools2D.area2(AScr, BScr, QScr) <= 0
```

then the whole line PQ and point C lie on different sides of the line AB, so that triangle ABC does not obscure line segment PQ. After this test for the triangle edge AB, we use similar tests for edges BC and CA.

Note that P and Q can both lie outside triangle ABC, while PQ intersects this triangle. This explains the above test, which at first may look quite complicated. Unfortunately, the current test does not cover all cases in which PQ lies outside triangle ABC, as you can see in Fig. C.5.

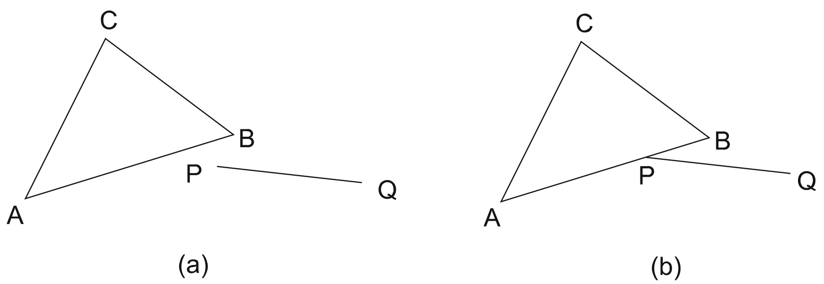


Fig. C.4 Test 4: P and Q on a side of AB different from that of C

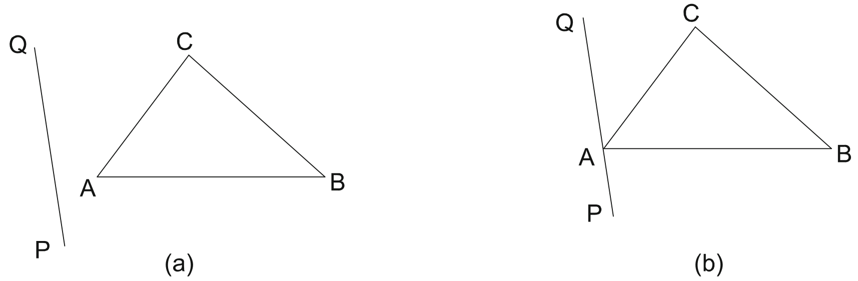


Fig. C.5 Test 5: triangle ABC on one side of PQ

Test 5 (2D; Fig. C.5)

Triangle ABC does not obscure PQ if the points A, B and C lie on the same side of the infinite line through P and Q. We determine if this is the case using a test that is similar to the previous one:

$$(PQA \leq 0 \text{ and } PQB \leq 0 \text{ and } PQC \leq 0) \text{ or}$$
$$(PQA \geq 0 \text{ and } PQB \geq 0 \text{ and } PQC \geq 0)$$

where PQA, obtained as

```
double PQA = Tools2D.area2(PScr, QScr, AScr);
```

denotes twice the area of triangle PQA, preceded by a minus sign if the point sequence P, Q, A is clockwise. The variables PQB and PQC have similar meanings.

Test 6 (3D; Fig. C.6)

If neither P nor Q lies behind the plane through A, B and C, triangle ABC does not obscure PQ. This test deals with those line segments PQ for which Test 3 failed because the farther one of the points P and Q does *not* lie nearer than the nearest of A, B, C, while P and Q nevertheless lie on the same side of the (infinite) plane ABC as the viewpoint E. Figure C.6 illustrates this situation.

We now benefit from the fact that we have stored the normal vector $\mathbf{n} = (a, b, c)$ of plane ABC and the distance h between E and this plane. The equation of this plane is

$$ax + by + cz = h$$

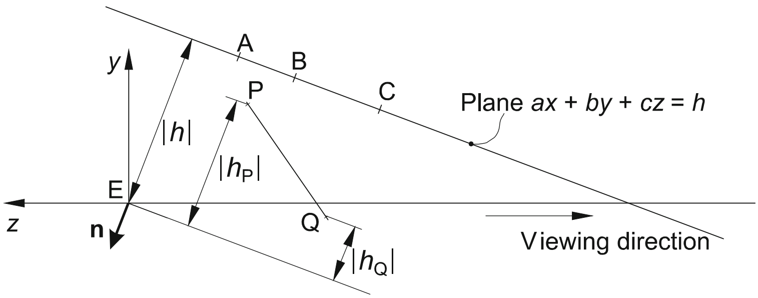


Fig. C.6 Test 6: neither P nor Q behind plane ABC

We compute

$$h_P = \mathbf{n} \times \mathbf{EP}$$
$$h_Q = \mathbf{n} \times \mathbf{EQ}$$

to perform the following test, illustrated by Fig. C.6:

$$|h_P| \le |h| \text{ and } |h_Q| \le |h|$$

Since **n**, when starting at E, points away from the plane, the values of h_P, h_Q, like that of h, are negative, so that this test is equivalent to the following:

$$h_P \ge h \text{ and } h_Q \ge h$$

Test 7 (2D; Fig. C.7)

If (on the screen) neither P nor Q lies outside the triangle ABC, and the previous tests were indecisive, PQ lies behind this triangle and is therefore completely invisible. The *Tools2D* method *insideTriangle*, discussed in Sect. 2.3, makes this test easy to program:

```
boolean pInside = Tools2D.insideTriangle(aScr, bScr, cScr, pScr);
boolean qInside = Tools2D.insideTriangle(aScr, bScr, cScr, qScr);
if (pInside && qInside) return;
```

The partial results of this test, stored in the boolean variables *pInside* and *qInside*, will be useful in Tests 8 and 9 in the case that this test fails.

Test 8 (3D; Fig. C.8)

If P is nearer than the plane of triangle ABC and, on the screen, P lies inside triangle ABC, this triangle does not obscure PQ. The same for Q.

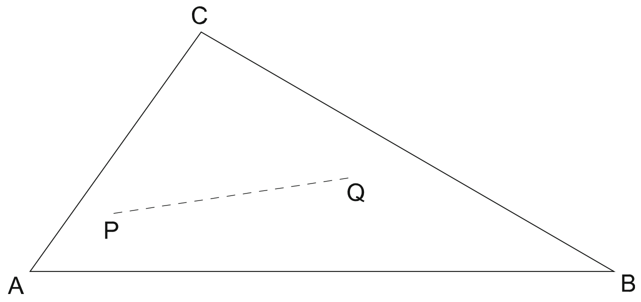


Fig. C.7 Test 7: PQ behind triangle ABC

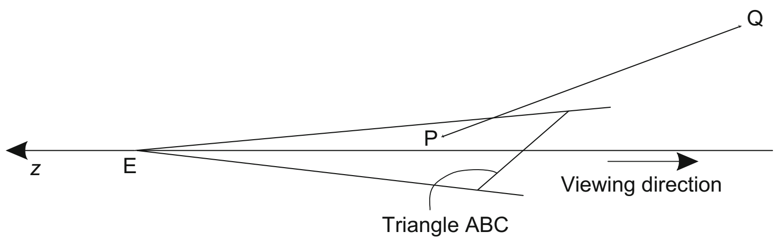


Fig. C.8 Test 8: P nearer than plane ABC

This test relies on the fact that no line segment PQ intersects any triangle. This test is easy to perform now that the variables h_P and h_Q, computed in Test 6, and *pInside* and *qInside*, computed in Test 7, are available. Using also the abbreviations *pNear* for $h_P > h$ and *qNear* for $h_Q > h$, we conclude that triangle ABC does not obscure PQ if the following is true:

```
pNear && pInside || qNear && qInside
```

Test 9 (3D; Fig. C.9)

Although it might seem that, after Test 8, all cases with PQ not being obscured have been dealt with, this is not the case, as Fig. C.9 illustrates. In this example, Q lies behind the plane ABC, and, on the screen, PQ intersects ABC in the points I and J, but PQ is nevertheless completely visible (as far as triangle ABC is concerned). This is because both points of intersection, I and J, lie in front of (that is, nearer than) triangle ABC. We take three steps to deal with this situation:

1. We compute (the 2D projections of) I and J on the screen.
2. We compute the z-values of I and J by linear interpolation of $1/z$ (see Sect. 6.4 and Appendix A).

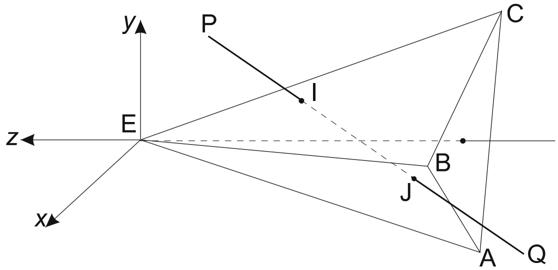


Fig. C.9 Test 9: I and J in front of plane ABC

3. To determine if I and J lie in front of the plane ABC, we compute how far I and J lie away in the direction towards this plane.

We will now discuss these three steps in more detail, starting with step 1, which we deal with as a 2D problem, writing, for example, P for what is actually the projection P′ of P on the screen.

Since we do not know in advance which of the three triangle edges AB, BC and CA may intersect PQ, we try all three by rotating the points A, B and C. Here we deal only with AB. Suppose that, on the screen, the infinite line PQ intersects the infinite line AB in point I, where

$$\mathbf{PI} = \lambda\mathbf{PQ} \tag{C.1}$$

$$\mathbf{AI} = \mu\mathbf{AB} \tag{C.2}$$

Then point I belongs to the line *segments* PQ and AB if and only if

$$0 \leq \lambda \leq 1 \text{ and } 0 \leq \mu \leq 1$$

While rotating A, B and C, we may find two points of intersection, with λ and μ satisfying these restrictions. We then denote the values λ of these points by λ_{min} and λ_{max}, and the corresponding points with I and J, so that we have

$$0 \leq \lambda_{\text{min}} < \lambda_{\text{max}} \leq 1$$

$$\mathbf{PI} = \lambda_{\text{min}}\mathbf{PQ}$$

$$\mathbf{PJ} = \lambda_{\text{max}}\mathbf{PQ}$$

As for the actual computation of λ and μ, let us use the vectors $\mathbf{u} = \mathbf{PQ}$ and $\mathbf{v} = \mathbf{AB}$. It then follows from Eqs. (C.1 and C.2) that we can express point I as the left- and right-hand sides of the following equation:

$$\mathbf{P} + \lambda\mathbf{u} = \mathbf{A} + \mu\mathbf{v}$$

Writing $\mathbf{w} = \mathbf{PA}$ (= $\mathbf{A} - \mathbf{P}$), we can replace this with

$$\lambda\mathbf{u} - \mu\mathbf{v} = \mathbf{w}$$

As usual, we write $\mathbf{u} = (u_1, u_2)$ and so on, which expands this vector equation to the following set of simultaneous linear equations:

$$u_1\lambda - \mathrm{v}_1\mu = w_1$$

$$u_2\lambda - \mathrm{v}_2\mu = w_2$$

Solving this system of equations, we obtain

$$\lambda = \frac{v_1 w_2 - v_2 w_1}{u_2 v_1 - u_1 v_2}$$
$$\mu = \frac{u_1 w_2 - u_2 w_1}{u_2 v_1 - u_1 v_2}$$

It goes without saying that this applies only if the denominator in these expressions is nonzero; otherwise PQ and AB are parallel, so that there are no points of intersection.

Having found the points I and J on the screen, we turn to step 2, to compute their z-coordinates. We do this by linear interpolation of $1/z$ on the segment PQ. Using the values $\lambda_{\min}$ and $\lambda_{\max}$, which we have just found, and referring to Eq. (C.1), we can write

$$\frac{1}{z_\mathrm{I}} = \frac{1}{z_\mathrm{P}} + \lambda_{\min}\left(\frac{1}{z_\mathrm{Q}} - \frac{1}{z_\mathrm{P}}\right) = \frac{\lambda_{\min}}{z_\mathrm{Q}} + \frac{1 - \lambda_{\min}}{z_\mathrm{P}}$$

$$\frac{1}{z_\mathrm{J}} = \frac{1}{z_\mathrm{P}} + \lambda_{\max}\left(\frac{1}{z_\mathrm{Q}} - \frac{1}{z_\mathrm{P}}\right) = \frac{\lambda_{\max}}{z_\mathrm{Q}} + \frac{1 - \lambda_{\max}}{z_\mathrm{P}}$$

Refer to the discussion in Sect. 6.4 and Appendix A about the reason why we should use $1/z$ instead of simply z in this type of linear interpolation.

Finally, we proceed to step 3, to determine whether the points I and J lie in front of the plane through the points A, B and C. Recall that the equation of this plane is

$$ax + by + cz = h$$

where h is negative, and that its normal vector (with length 1) is

$$\mathbf{n} = (a, b, c)$$

We compute the value h_I, which is similar to h_P, discussed in Test 6 and illustrated by Fig. C.6, as

$$h_\mathrm{I} = \mathbf{EI} \times \mathbf{n} = ax_\mathrm{I} + by_\mathrm{I} + cz_\mathrm{I}$$

After computing h_J similarly, we can now test if I and J lie in front of the plane ABC (so that triangle ABC does not obscure PQ) in the following way:

$$h_\mathrm{I} > h \text{ and } h_\mathrm{J} > h$$

In the above discussion, we considered two distinct points I and J in which, on the screen, PQ intersects edges of triangle ABC. As we have seen, PI and JQ were visible, as far as triangle ABC is concerned, but IJ may be obscured by triangle ABC. Actually, there may be only one point, I or J, to deal with. If, again on the screen, P lies outside triangle ABC and Q inside it, there is only point I to consider. In this case triangle ABC may obscure part IQ of line segment PQ. If it does, the remaining triangles are only to be applied to PI. Similarly, if, on the screen, P lies inside and Q outside triangle ABC, this triangle may obscure part PJ of PQ, and, if so, the remaining triangles are only to be applied to JQ.

Recursive Calls

If all the above tests fail, the most interesting (and time consuming) case applies: PQ is neither completely visible nor completely hidden. Fortunately, we have just computed the points of intersection I and J, and we know that triangle ABC obscures the segment IJ, while the other two segments, PI and QJ are visible, as far as triangle ABC is concerned. We therefore apply the method *lineSegment* recursively to the latter two segments. Actually, the recursive call for PI applies only to the case that, on the screen, P lies outside triangle ABC and $\lambda_{\min}$ (see Test 9) is greater than zero. Analogously, the recursive call for QJ applies only if Q lies outside that triangle and $\lambda_{\max}$ is less than 1.

The Arguments of the **lineSegment** *Method*

In the *paint* method of the class *CvHLines*, we may be inclined to write the call to the method *lineSegment* in a very simple form, such as

```
lineSegment(g, iP, iQ);
```

where *g* is the graphics context and *iP* and *iQ* are the vertex numbers of the vertices P and Q. However, in the recursive calls just discussed, we have two new points I and J, for which there are no vertex numbers, so that this does not work. On the other hand, omitting the vertex numbers altogether would deprive us of Test 2 in its current efficient form, in which we determine if PQ is one of the edges of triangle ABC. We therefore decide to supply P and Q both as *Point3D* objects (containing the eye-coordinates of P and Q) and as vertex numbers if this is possible; if it is not, we use −1 instead of a vertex number. When we recursively call *lineSegment*, the screen coordinates of P and Q are available. If we did not supply these as arguments, it would be necessary to compute them inside *lineSegment* once again. To avoid such superfluous actions, we also supply the screen coordinates of P and Q as arguments as *Point2D* objects. Finally, it would be a waste of time if the recursive calls would again be applied to all triangles. After all, we know that PI and PJ are not obscured by the triangles that we have already dealt with. We therefore also use the parameter *iStart*, indicating the start position in the array of triangles that is to be used. This explains that the method *lineSegment* has as many as eight parameters, as its heading shows:

```
void lineSegment(Graphics g, Point3D p, Point3D q,
Point2D PScr, Point2D QScr, int iP, int iQ, int iStart)
```

The complete method *lineSegment* can be found in class *CvHLines*, listed in Sect. C.3.

C.2. HP-GL Output and Class *HLines*

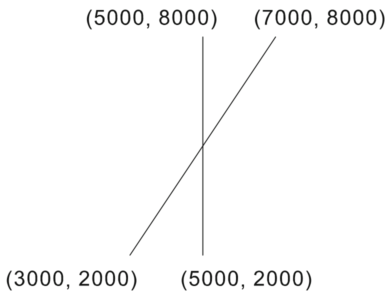
Besides graphics output on the screen, it is sometimes desired to produce output files containing the same results. An easy way to realize this for line drawings is by using the file format known as *HP-GL*, which stands for *Hewlett-Packard Graphics Language*. We will use only a very limited number of HP-GL commands:

IN: Initialize
SP: Set pen
PU: Pen up
PD: Pen down
PA: Plot absolute

We think of drawing by moving a pen, which can be either on or above a sheet of paper. These two cases are distinguished by the commands *PD* and *PU*. The *PA* command is followed by a coordinate pair *x*, *y*, each as a four-digit integer in the range 0000–9999. This coordinate pair indicates a point that the pen will move to. The origin (0000, 0000) lies in the bottom-left corner. For example, the following HP-GL file draws a capital letter *X* in italic, shown in Fig. C.10:

```
IN;SP1;
PU;PA5000,2000;PD;PA5000,8000;
PU;PA3000,2000;PD;PA7000,8000;
```

Fig. C.10 Letter X in italic



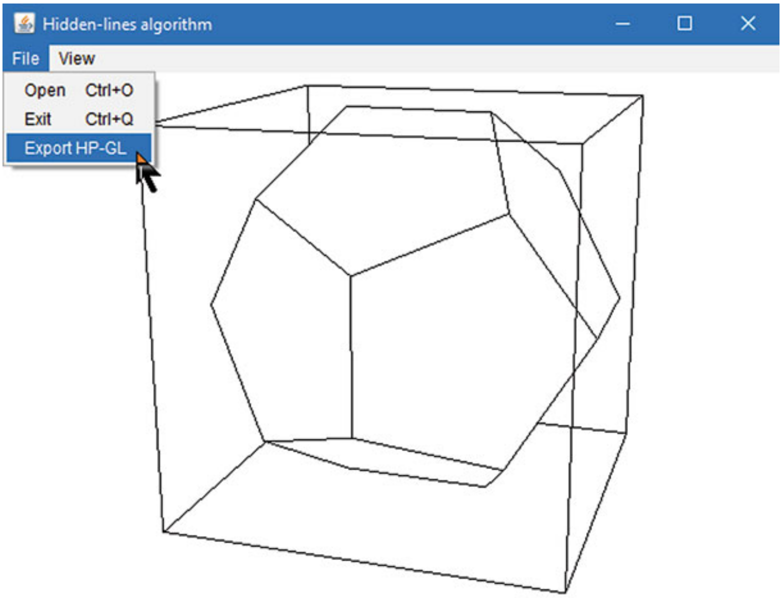


Fig. C.11 The export HP-GL menu item

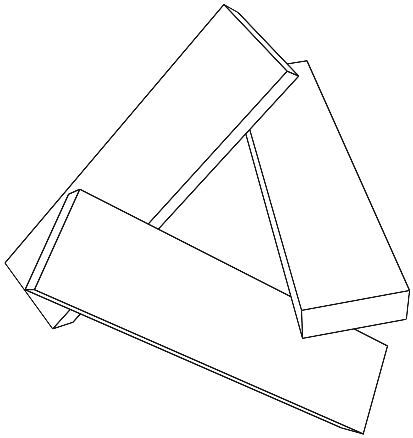
Originally designed for pen plotters, this file format is easy to understand, and it is accepted by several well-known packages, such as *CorelDraw*. It can also be converted to other file formats, such as Scalable Vector Graphics (*xxx.svg*) and Windows Meta File (*xxx.wmf*) by using a conversion utility, such as *UniConvertor* or *HP2XX*. Then we can use a vector-oriented drawing package, such as *Inkscape* or *CorelDraw* to edit and enhance the drawing. This was done to add text to the current example, resulting in Fig. C.10.

Many illustrations showing (black/white) line drawings of 3D objects in this book were made in this way. This is possible because our program *HLines.java*, listed in the next subsection, can generate HP-GL files, as Fig. C.11 shows by highlighting the menu item *Export HP-GL*.

This example also shows the use of individual line segments, as discussed in Sect. 5.5. The solid object inside the wire-frame cube is a dodecahedron, which is discussed in greater detail in Sect. D.1 of Appendix D.

Figure C.12 shows that our hidden-line algorithm can correctly render the three beams used in Sect. 6.5 to demonstrate the Z-buffer algorithm. As we have seen in Sect. 6.4, the painter's algorithm fails in such cases.

Fig. C.12 Three beams



Implementation

Our hidden-line program's *main* method generates an object of class *Fr3D* and also of class *CvHLines*, as the class *HLines* shows:

```
// HLines.java: Perspective drawing with hidden-line elimination.
// Uses: HPGL (Section C.2), CvHLines (Section C.3),
//       Fr3D, Polygon3D, Obj3D, Input, Canvas3D,
//       Tria (Section 5.6), Point2D (Section 1.4),
//       Point3D (Section 3.9), Tools2D (Section 2.3).
import java.awt.*;

public class HLines extends Frame {

    public static void main(String[] args) {
       new Fr3D(args.length > 0 ? args[0] : null, new CvHLines(),
           "Hidden-lines algorithm");
    }
}
```

Class *Fr3D*, introduced in Sect. 5.6, contains the following fragment to produce the command *Export HP-GL* if an object of this class is generated by the *HLines* program:

```
if (hiddenLines){
    exportHPGL = new MenuItem("Export HP-GL");
    mF.add(exportHPGL); // mF defined in Fr3D
    exportHPGL.addActionListener(this);
}
```

The if-clause checks that this command is only available for the hidden-line algorithm, not for the *Painter* and *ZBuf* algorithms. Notice the title *Hidden-lines algorithm*, used as the third argument of the *Fr*3*D* constructor, so this text appears in the title bar of the window shown in Fig. C.11. The implementation of the hidden-line algorithm discussed in the previous section can be found in the class *CvHLines*. In view of the extent of this class, it is not listed here but rather as the file *CvHLines.java* in the next section.

The class *Fr*3*D*, listed in Sect. 5.6, shows that the use of the *Eport HPGL* menu command leads to the generation of an object of class *HPGL*. This class is listed below:

```
// HPGL.java: Class for export of HP-GL files.
import java.io.*;

class HPGL {
   FileWriter fw;

   HPGL(Obj3D obj) {
      String plotFileName = "", fName = obj.getFName();
      for (int i = 0; i < fName.length(); i++) {
         char ch = fName.charAt(i);
         if (ch == '.') break;
         plotFileName += ch;
      }
      plotFileName += ".plt";
      try {
         fw = new FileWriter(plotFileName);
         fw.write("IN;SP1;\n");
      }
      catch (IOException ioe) {
      }
   }

   void write(String s) {
   try {fw.write(s); fw.flush();} catch (IOException ioe) {}
   }
}
```

This shows that a generated HP-GL file has the file extension *.plt*. For example, with an input file *cube.dat*, the HP-GL file is *cube.plt*, and it is written in the same directory as the input file. The above method *write* is called in the method *drawLine* of class *CvHLines*, listed in the next section.

C.3. Class *CvHLines*

As we have seen a moment ago, the following class is used in program *HLines.java*:

```
// CvHLines.java: Used in the file HLines.java.
import java.awt.*;
import java.util.*;

class CvHLines extends Canvas3D {
   private int maxX, maxY, centerX, centerY, nTria, nVertices;
   private Obj3D obj;
   private Point2D imgCenter;
   private Tria[] tr;
   private HPGL hpgl;
   private int[] refPol;
   private Vector<Integer> connect[];
   private double hLimit;
   private Vector<Polygon3D> polyList;
   private float maxScreenRange;

   Obj3D getObj() {return obj;}
   void setObj(Obj3D obj) {this.obj = obj;}
   void setHPGL(HPGL hpgl) {this.hpgl = hpgl;}

   public void paint(Graphics g) {
      if (obj == null)
         return;
      Vector<Polygon3D> polyList = obj.getPolyList();
      if (polyList == null) return;
      int nFaces = polyList.size();
      if (nFaces == 0) return;
      Dimension dim = getSize();
      maxX = dim.width - 1; maxY = dim.height - 1;
      centerX = maxX / 2; centerY = maxY / 2;
      // ze-axis towards eye, so ze-coordinates of
      // object points are all negative. Since screen
      // coordinates x and y are used to interpolate for
      // the z-direction, we have to deal with 1/z instead
      // of z. With negative z, a small value of 1/z means
      // a small value of |z| for a nearby point.

      // obj is a java object that contains all data,
      // with w, e and vScr parallel (with vertex numbers
      // as index values):
```

```
// - Vector w (with Point3D elements)
// - Array e (with Point3D elements)
// - Array vScr (with Point2D elements)
// - Vector polyList (with Polygon3D elements)

// Every Polygon3D value contains:
// - Array 'nrs' for vertex numbers (n elements)
// - Values a, b, c, h for the plane ax+by+cz=h.
// - Array t (with n-2 elements of type Tria)

// Every Tria value consists of the three vertex
// numbers A, B and C.
maxScreenRange = obj.eyeAndScreen(dim);
imgCenter = obj.getImgCenter();
obj.planeCoeff(); // Compute a, b, c and h.

hLimit = -1e-6 * obj.getRho();
buildLineSet();

// Construct an array of triangles in each polygon and count
// the total number of triangles.
nTria = 0;
for (int j = 0; j < nFaces; j++) {
    Polygon3D pol = polyList.elementAt(j);
    if (pol.getNrs().length > 2 && pol.getH() <= hLimit) {
        pol.triangulate(obj);
        nTria += pol.getT().length;
    }
}
tr = new Tria[nTria]; // Triangles of all polygons
refPol = new int[nTria]; // tr[i] belongs to refPol[i]
int iTria = 0;

for (int j = 0; j < nFaces; j++) {
  Polygon3D pol = polyList.elementAt(j);
  Tria[] t = pol.getT(); // Triangles of one polygon
  if (pol.getNrs().length > 2 && pol.getH() <= hLimit) {
     for (int i = 0; i < t.length; i++) {
        Tria tri = t[i];
        tr[iTria] = tri;
        refPol[iTria++] = j;
     }
   }
 }
```

```
      Point3D[] e = obj.getE();
      Point2D[] vScr = obj.getVScr();
      for (int i = 0; i < nVertices; i++) {
         for (int j = 0; j < connect[i].size(); j++) {
            int jj = connect[i].elementAt(j).intValue();
            lineSegment(g,e[i],e[jj],vScr[i],vScr[jj],i,jj,0);
         }
      }
      hpgl = null;
   }

   private void buildLineSet() {
      // Build the array 'connect', where connect[i] is a
      // Vector<Integer> containing all vertex numbers j,
      // such that (i, connect[i].elementAt(j).intValue())
      // is an edge of the 3D object.
      polyList = obj.getPolyList();
      nVertices = obj.getVScr().length;
      connect = new Vector[nVertices];
      for (int i=0; i<nVertices; i++)
        connect[i] = new Vector<Integer>();
      int nFaces = polyList.size();

      for (int j = 0; j < nFaces; j++) {
         Polygon3D pol = polyList.elementAt(j);
         int[] nrs = pol.getNrs();
         int n = nrs.length;
         if (n > 2 && pol.getH() > 0) continue;
         int ii = Math.abs(nrs[n - 1]);
         for (int k = 0; k < n; k++) {
            int jj = nrs[k];
            if (jj < 0) jj = -jj; // abs
            else {
              int i1 = Math.min(ii, jj),
                  j1 = Math.max(ii, jj);
              Integer j1Int = new Integer(j1);
              if (connect[i1].indexOf(j1Int) == -1)
                 connect[i1].addElement(j1Int);
            }
            ii = jj;
         }
      }
   }
```

```
int iX(float x) {return Math.round(centerX + x - imgCenter.x);}
int iY(float y) {return Math.round(centerY - y + imgCenter.y);}

private String toString(float t) {
// From screen device units (pixels) to HP-GL units (0-10000):
   int i = Math.round(5000 + t * 9000 / maxScreenRange);
   String s = "";
   int n = 1000;
   for (int j = 3; j >= 0; j--) {
     s += i / n;
     i %= n;
     n /= 10;
   }
   return s;
}

private String hpx(float x) {return toString(x - imgCenter.x);}
private String hpy(float y) {return toString(y - imgCenter.y);}

private void drawLine(Graphics g, float x1, float y1,
     float x2, float y2) {
   if (x1 != x2 || y1 != y2) {
     g.drawLine(iX(x1), iY(y1), iX(x2), iY(y2));
     if (hpgl != null) {
       hpgl.write("PU;PA" + hpx(x1) + "," + hpy(y1) + ";");
       hpgl.write("PD;PA" + hpx(x2) + "," + hpy(y2) + ";\n");
     }
   }
}

private void lineSegment(Graphics g, Point3D p, Point3D q,
      Point2D pScr, Point2D qScr, int iP, int iQ, int iStart) {
   double u1 = qScr.x - pScr.x, u2 = qScr.y - pScr.y;
   double minPQx = Math.min(pScr.x, qScr.x);
   double maxPQx = Math.max(pScr.x, qScr.x);
   double minPQy = Math.min(pScr.y, qScr.y);
   double maxPQy = Math.max(pScr.y, qScr.y);
   double zP = p.z, zQ = q.z; // p and q give eye-coordinates
   double minPQz = Math.min(zP, zQ);
   Point3D[] e = obj.getE();
   Point2D[] vScr = obj.getVScr();
   for (int i = iStart; i < nTria; i++) {
     Tria t = tr[i];
     int iA = Math.abs(t.iA), iB = Math.abs(t.iB),
         iC = Math.abs(t.iC);
     Point2D aScr = vScr[iA], bScr = vScr[iB], cScr = vScr[iC];
```

```
// 1. Minimax test for x and y screen coordinates:
if (maxPQx <= aScr.x && maxPQx <= bScr.x && maxPQx <= cScr.x ||
    minPQx >= aScr.x && minPQx >= bScr.x && minPQx >= cScr.x ||
    maxPQy <= aScr.y && maxPQy <= bScr.y && maxPQy <= cScr.y ||
    minPQy >= aScr.y && minPQy >= bScr.y && minPQy >= cScr.y)
   continue; // This triangle does not obscure PQ.

// 2. Test if PQ is an edge of ABC:
if ((iP == iA || iP == iB || iP == iC) &&
    (iQ == iA || iQ == iB || iQ == iC))
   continue; // This triangle does not obscure PQ.

// 3. Test if PQ is clearly nearer than ABC:
double zA = e[iA].z, zB = e[iB].z, zC = e[iC].z;
if (minPQz >= zA && minPQz >= zB && minPQz >= zC)
    continue; // This triangle does not obscure PQ.

// 4. Do P and Q (in 2D) lie in a half plane defined
// by line AB, on the side other than that of C?
// Similar for the edges BC and CA.
double eps = 0.1; // Relative to numbers of pixels
if (Tools2D.area2(aScr, bScr, pScr) < eps &&
    Tools2D.area2(aScr, bScr, qScr) < eps ||
    Tools2D.area2(bScr, cScr, pScr) < eps &&
    Tools2D.area2(bScr, cScr, qScr) < eps ||
    Tools2D.area2(cScr, aScr, pScr) < eps &&
    Tools2D.area2(cScr, aScr, qScr) < eps)
   continue; // This triangle does not obscure PQ.

// 5. Test (2D) if A, B and C lie on the same side
// of the infinite line through P and Q:
double pqa = Tools2D.area2(pScr, qScr, aScr);
double pqb = Tools2D.area2(pScr, qScr, bScr);
double pqc = Tools2D.area2(pScr, qScr, cScr);

if (pqa < +eps && pqb < +eps && pqc < +eps ||
    pqa > -eps && pqb > -eps && pqc > -eps)
   continue; // This triangle does not obscure PQ.

// 6. Test if neither P nor Q lies behind the
// infinite plane through A, B and C:
int iPol = refPol[i];
Polygon3D pol = polyList.elementAt(iPol);
```

```
double a = pol.getA(), b = pol.getB(), c = pol.getC(),
       h = pol.getH(), eps1 = 1e-5 * Math.abs(h),
       hP = a * p.x + b * p.y + c * p.z,
       hQ = a * q.x + b * q.y + c * q.z;
if (hP > h - eps1 && hQ > h - eps1)
   continue; // This triangle does not obscure PQ.

// 7. Test if both P and Q behind triangle ABC:
boolean pInside =
    Tools2D.insideTriangle(aScr, bScr, cScr, pScr);
boolean qInside =
    Tools2D.insideTriangle(aScr, bScr, cScr, qScr);
if (pInside && qInside)
    return; // This triangle obscures PQ.

// 8. If P nearer than ABC and inside, PQ visible;
// the same for Q:
double h1 = h + eps1;
if (hP > h1 && pInside || hQ > h1 && qInside)
     continue; // This triangle does not obscure PQ.

// 9. Compute the intersections I and J of PQ
// with ABC in 2D.
// If, in 3D, such an intersection lies in front of
// ABC, this triangle does not obscure PQ.
// Otherwise, the intersections lie behind ABC and
// this triangle obscures part of PQ:
double lambdaMin = 1.0, lambdaMax = 0.0;
for (int ii = 0; ii < 3; ii++) {
    double v1 = bScr.x - aScr.x, v2 = bScr.y - aScr.y,
           w1 = aScr.x - pScr.x, w2 = aScr.y - pScr.y,
           denom = u2 * v1 - u1 * v2;
    if (denom != 0) {
       double mu = (u1 * w2 - u2 * w1) / denom;
        // mu = 0 gives A and mu = 1 gives B.
        if (mu > -0.0001 && mu < 1.0001) {
           double lambda = (v1 * w2 - v2 * w1) / denom;
           // lambda = PI/PQ
           // (I is point of intersection)
           if (lambda > -0.0001 && lambda < 1.0001) {
             if (pInside != qInside &&
             lambda > 0.0001 && lambda < 0.9999) {
             lambdaMin = lambdaMax = lambda;
             break; // Only one point of intersection
           }
```

```
                if (lambda < lambdaMin) lambdaMin = lambda;
                if (lambda > lambdaMax) lambdaMax = lambda;
             }
          }
       }
      Point2D temp = aScr; aScr = bScr; bScr = cScr; cScr = temp;
    }
    float d = obj.getD();
    if (!pInside && lambdaMin > 0.001) {
      double iScrx = pScr.x + lambdaMin * u1,
             iScry = pScr.y + lambdaMin * u2;
      // Back from screen to eye coordinates:
      double zI = 1 / (lambdaMin / zQ + (1 - lambdaMin) / zP),
             xI = -zI * iScrx / d, yI = -zI * iScry / d;
      if (a * xI + b * yI + c * zI > h1)
         continue; // This triangle does not obscure PQ.

      Point2D iScr = new Point2D((float) iScrx, (float) iScry);
      if (Tools2D.distance2(iScr, pScr) >= 1.0)
          lineSegment(g, p, new Point3D(xI, yI, zI), pScr, iScr,
                iP, -1, i + 1);
    }
    if (!qInside && lambdaMax < 0.999) {
       double jScrx = pScr.x + lambdaMax * u1,
              jScry = pScr.y + lambdaMax * u2;
       double zJ = 1 / (lambdaMax / zQ + (1 - lambdaMax) / zP),
              xJ = -zJ * jScrx / d, yJ = -zJ * jScry / d;
       if (a * xJ + b * yJ + c * zJ > h1)
          continue; // This triangle does not obscure PQ.
       Point2D jScr = new Point2D((float) jScrx, (float) jScry);
       if (Tools2D.distance2(jScr, qScr) >= 1.0)
         lineSegment(g, q, new Point3D(xJ, yJ, zJ),
                     qScr, jScr, iQ, -1, i + 1);
    }
    return;
    // if no continue-statement has been executed
  }
  drawLine(g, pScr.x, pScr.y, qScr.x, qScr.y);
  // No triangle obscures PQ.
 }
}
```

Appendix D: Several 3D Objects

This appendix demonstrates how to generate data files for other interesting 3D objects by special programs, in a similar fashion as the cylinder example in Sect. 5.8. The generated files are accepted by the programs *HLines.java*, *Painter.java* and *ZBuf.java* (see Chap. 6).

D.1. Platonic Solids

We will first discuss the generation of 3D files for five well-known objects. Let us begin with two definitions.

If all edges of a polygon have the same length and any two edges meeting at a vertex include the same angle, the polygon is said to be *regular*. If all bounding faces of a polyhedron are regular polygons, which are congruent (that is, which have exactly the same shape), that polyhedron is referred to as a *regular polyhedron* or *platonic solid*. There are only five essentially different platonic solids; their names and their numbers of faces, edges and vertices are listed below:

Platonic solid	Faces	Edges	Vertices
Tetrahedon	4	6	4
Cube (= hexahedron)	6	12	8
Octahedron	8	12	6
Dodecahedron	12	30	20
Icosahedron	20	30	12

Note that what we call a *tetrahedron*, a *hexahedron*, and so on, should actually be referred to as *regular tetrahedron*, *regular hexahedron*, etc., but since in this section we are only dealing with regular polyhedra, we omit the word *regular* here.

L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2

The above numbers of faces, edges and vertices satisfy Euler's theorem, which also applies to non-regular polyhedra:

$$Faces + Vertices = Edges + 2 \tag{D.1}$$

Tetrahedron

An elegant way of constructing a tetrahedron is by using the diagonals of the six faces of a cube as the edges of the tetrahedron, as Fig. D.1a shows.

Using a cube with edges of length 2 and with its center as the origin of the coordinate system, we can easily write down the contents of the data file, say, *tetra.dat*, for the tetrahedron proper (without the surrounding cube):

```
1 1 -1 -1
2 -1 1 -1
3 1 1 1
4 -1 -1 1
Faces:
1 2 3.
2 4 3.
1 4 2.
1 3 4.
```

Supplying this file to program *HLines.java* results in a poor representation of a tetrahedron, identical to what we obtain if we omit the dashed line (and the vertex numbers) in Fig. D.1b. Most people would fail to recognize that result as a 3D object, unless Fig. D.1a is also given. The surrounding cube in the latter figure can be obtained by adding the vertices 5, 6, 7 and 8 to the first part of the above input file and 12 lines of vertex-pair numbers, for the cube edges, to the second.

A much better representation is Fig. D.1c, in which the tetrahedron has been rotated about the edge 1-2, in such a way that the face 1-2-3 becomes horizontal. The angle of this rotation is equal to

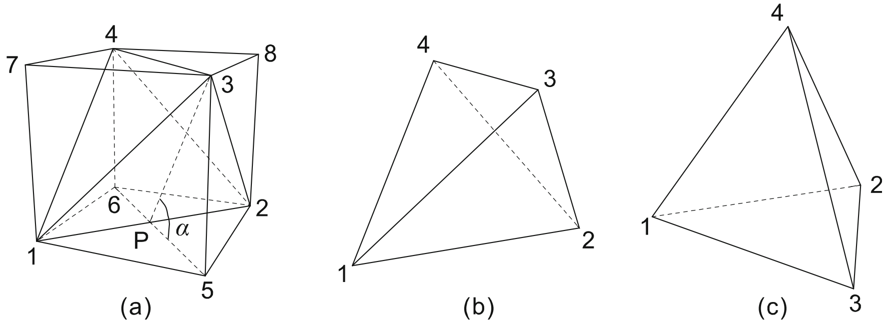


Fig. D.1 A tetrahedron: (**a**) inside a cube; (**b**) cube omitted; (**c**) after rotation

$$\alpha = \arctan\sqrt{2}$$

As Fig. D.1a shows, α is the angle included by the edges P-5 and P-3 in the right-angled triangle P-5-3, where P is the point of intersection of the lines 1-2 and 5-6. Since the cube edge 3-5 has length 2 and the line segment P-5, being half the diagonal 6–5 of the bottom plane, has the length $\sqrt{2}$, we have $\tan \alpha = 2/\sqrt{2} = \sqrt{2}$, which explains the above value of α.

Fortunately, we have already developed a general and useful class, *Rota3D*, for 3D rotations. We used it in Sect. 3.9 in the program *Rota3DTest.java* to rotate a cube about a given axis AB through an angle of 180°. This time, the vertices 1 and 2 will act as points A and B, while rotation will take place through the angle α we have just been discussing. We will use the file *tetra.dat*, listed above, as input, to derive the file *tetra1.dat*, describing the rotated tetrahedron, from it. Note that the classes *Point3D* and *Rota3D*, both defined in Sect. 3.9, and the class *Input* of Sect. 5.6 must be available (in the form of *.class* files) in the current directory.

```
// Rota3DTetra.java: Rotating a tetrahedron that
//   has horizontal top and bottom edges, in such
//   a way that it obtains a horizontal bottom face.
//   Uses: Point3D, Rota3D (Section 3.9), Input (Section 5.6).
import java.io.*;

public class Rota3DTetra {
   public static void main(String[] args) throws IOException {
     // Specify AB as directed axis of rotation and alpha as the
     // rotation angle:
     Point3D A = new Point3D(1, -1, -1), B = new Point3D(-1, 1, -1);
     double alpha = Math.atan(Math.sqrt(2));
     Rota3D.initRotate(A, B, alpha);
     Point3D P = new Point3D(0, 0, 0);
     Input inp = new Input("tetra.dat");
     if (inp.fails()) {
        System.out.println("Supply file tetra.dat, see Section D.1");
        System.exit(0);
     }
     FileWriter fw = new FileWriter("tetra1.dat");
     for (;;) {
        int i = inp.readInt();
        if (inp.fails())
           break;
        P.x = inp.readFloat();
        P.y = inp.readFloat();
        P.z = inp.readFloat();
```

```
            Point3D P1 = Rota3D.rotate(P);
            fw.write(i + " " + P1.x + " " + P1.y + " " + P1.z + "\r\n");
         }
         inp.clear();
         // Copy the rest of file tetra.dat to tetra1.dat:
         for (;;) {
            char ch = inp.readChar();
            if (inp.fails()) break;
            fw.write(ch);
         }
         fw.close();
      }
}
```

This program changes only the coordinates of the four vertices. With the above input file, *tetra.dat*, the resulting output file, *tetra*1*.dat*, is listed below:

```
1 1.0 -1.0 -1.0
2 -1.0 1.0 -1.0
3 1.7320508 1.7320508 -1.0
4 0.57735026 0.57735026 1.309401
Faces:
1 2 3.
2 4 3.
1 4 2.
1 3 4.
```

Since we now have $z = -1.0$ for the vertices 1, 2 and 3, we see that triangle 1-2-3 is now horizontal. This is illustrated by Fig. D.1c, which, except for the text and the dashed edge, is the result of applying the program *HLines.java* to the above file *tetra*1*.dat*.

Cube or Hexahedron

The well-known cube, also known as a hexahedron, is perhaps the most popular 3D object because of its simplicity. With the rather unusual vertex numbering of Fig. D.1a, this cube (with edges of length 2 and with the origin as its center) is described by the following file:

```
1  1 -1 -1
2 -1  1 -1
3  1  1  1
4 -1 -1  1
5  1  1 -1
```

```
6 -1 -1 -1
7  1 -1  1
8 -1  1  1
Faces:
1 5 3 7.
5 2 8 3.
2 6 4 8.
6 1 7 4.
7 3 8 4.
1 6 2 5.
```

Octahedron

An octahedron has eight faces, which are equilateral triangles. One way of constructing this platonic solid is by starting with a cube and using the principle of *duality* or *reciprocity*. The numbers of faces and vertices shown in the table

Platonic solid	Faces	Edges	Vertices
Cube (= hexahedron)	6	12	8
Octahedron	8	12	6

suggest that each face of a cube might be related to a vertex of an octahedron and vice versa. This is indeed the case: a cube and an octahedron are said to be *dual* or *reciprocal*, which implies that the centers of the faces of one can be used as the vertices of the other. Starting with a cube, we can simply use the centers of its six faces as the vertices of a octahedron. The following data file for an octahedron is based on such a cube with edges of length 2 and with O as its center:

```
1  1  0  0
2  0  1  0
3 -1  0  0
4  0 -1  0
5  0  0 -1
6  0  0  1
Faces:
1 2 6.
2 3 6.
3 4 6.
4 1 6.
2 1 5.
3 2 5.
4 3 5.
1 4 5.
```

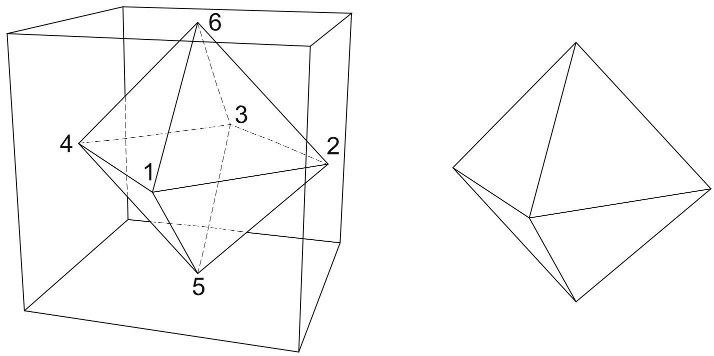


Fig. D.2 Octahedron

If we use this file as input for program *HLines.java*, we obtain the hexahedron as shown in Fig. D.2 on the right. On the left we see the way the vertices are numbered and the surrounding cube.

Icosahedron and Dodecahedron

Let us again consider the following numbers of faces and vertices:

Platonic solid	Faces	Edges	Vertices
Dodecahedron	12	30	20
Icosahedron	20	30	12

Since the number of faces of one of the two polyhedra is equal to the number of vertices of the other, a dodecahedron and an icosahedron are reciprocal polyhedra. Figure D.3 illustrates this. For example, vertex 1 of the dodecahedron has been constructed as the center of the face 1-2-3 of the icosahedron, vertex 2 of the dodecahedron as that of face 1-3-4 of the icosahedron, and so on. Since showing this dodecahedron in Fig. D.3 in its original position would have been confusing, the dodecahedron has been shifted to the right after its construction.

It follows that constructing an icosahedron is the only remaining problem: once the coordinates of its vertices are known, we can derive the dodecahedron from it.

Both polyhedra are based on regular pentagons, as you can see in Fig. D.3. In the icosahedron shown here, there are two horizontal pentagons: 2-3-4-5-6 and 7-8-9-10-11. They have both been constructed with their vertices on (horizontal) circles with radius 1. Curiously enough, these two horizontal pentagons lie a distance 1 apart, while the 'north pole' 1 and the 'south pole' 12 lie a distance $\sqrt{5}$ apart. A proof of these facts is omitted here, because it would be rather lengthy and only very loosely related to computer graphics. Interested readers are referred to books about geometry, such as that by Coxeter, listed in the Bibliography.

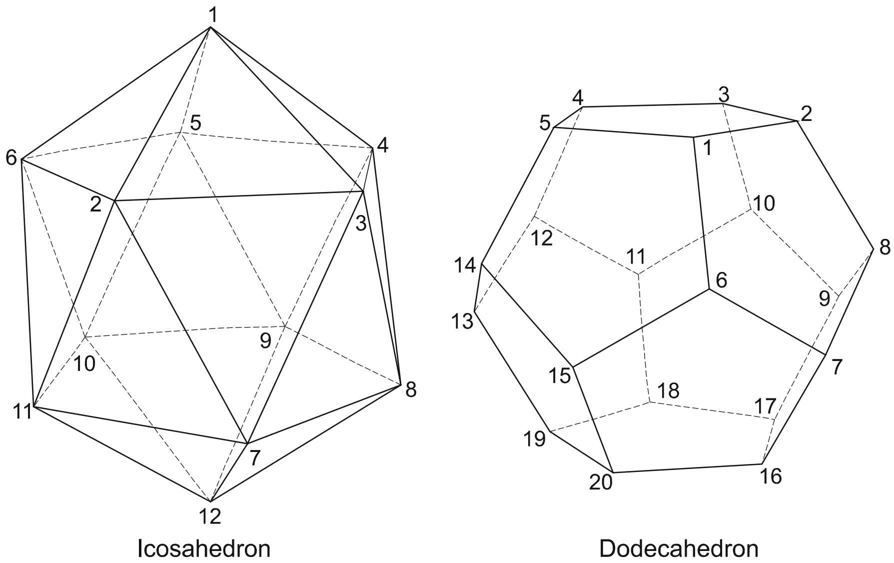


Fig. D.3 Icosahedron and dodecahedron as reciprocal polyhedra

With O as the center of the icosahedron, we can now define the coordinates of all 12 vertices of an icosahedron as follows:

i	x_i	y_i	z_i
1	0	0	$\frac{1}{2}\sqrt{5}$
2, 3, 4, 5, 6	$\cos\{(i-2)\times 72°\}$	$\sin\{(i-2)\times 72°\}$	0.5
7, 8, 9, 10 11	$\cos\{36° + (i-7)\times 72°\}$	$\sin\{36° + (i-7)\times 72°\}$	−0.5
12	0	0	$-\frac{1}{2}\sqrt{5}$

We can easily program the above formulas to obtain the numerical values of the vertex coordinate in question, as we will see shortly. These values are listed in the following 3D data file, in which the faces of the icosahedron are displayed on five text lines, instead of on 20 lines as generated by the program. Besides, the layout of the first part of the file has been improved for the sake of readability:

```
 1  0.0       0.0         1.118034
 2  1.0       0.0         0.5
 3  0.309017  0.95105654  0.5
 4 -0.809017  0.58778524  0.5
 5 -0.809017 -0.58778524  0.5
 6  0.309017 -0.95105654  0.5
 7  0.809017  0.58778524 -0.5
 8 -0.309017  0.95105654 -0.5
 9 -1.0       0.0        -0.5
10 -0.309017 -0.95105654 -0.5
```

```
11  0.809017 -0.58778524 -0.5
12  0.0       0.0        -1.118034
 Faces:
 1  2  3.  1  3 4.  1  4  5.  1  5  6.
 1  6  2.  2  7 3.  3  7  8.  3  8  4.
 4  8  9.  4  9 5.  5  9 10.  5 10  6.
 6 10 11.  6 11 2.  2 11  7. 12  8  7.
12  9  8. 12 10 9. 12 11 10. 12  7 11.
```

Dodecahedron

As mentioned before, we use the center of the first face, 1-2-3, of the icosahedron as vertex 1 of the dodecahedron. For example, we have

$$x_1(\text{of the dodecahedron}) = \frac{x_1 + x_2 + x_3\ (\text{all three of the icosahedron})}{3}$$

The faces in the above icosahedron file are to be read line by line, so that the center of face 1-3-4 provides vertex 2 of the dodecahedron, that of face 1-4-5 vertex 3, and so on. The second part of the dodecahedron file is more difficult. You can check it by comparing both polyhedra of Fig. D.3. For example, the vertices of the dodecahedron's face 1-6-7-8-2, in that order, are the centers of face 1 (1-2-3), face 6 (2-7-3), face 7 (3-8-7), face 8 (3-8-4) and face 2 (1-3-4) of the icosahedron. In the data file for the dodecahedron below, you can find this face 1-6-7-8-2 as the second one after the word *Faces*:

```
 1  0.436339    0.31701884  0.7060113
 2 -0.16666667  0.51294726  0.7060113
 3 -0.53934467  0.0         0.7060113
 4 -0.16666667 -0.51294726  0.7060113
 5  0.436339   -0.31701884  0.7060113
 6  0.7060113   0.51294726  0.16666667
 7  0.26967233  0.82996607 -0.16666667
 8 -0.26967233  0.82996607  0.16666667
 9 -0.7060113   0.51294726 -0.16666667
10 -0.872678    0.0         0.16666667
11 -0.7060113  -0.51294726 -0.16666667
12 -0.26967233 -0.82996607  0.16666667
13  0.26967233 -0.82996607 -0.16666667
14  0.7060113  -0.51294726  0.16666667
15  0.872678    0.0        -0.16666667
16  0.16666667  0.51294726 -0.7060113
17 -0.436339    0.31701884 -0.7060113
```

```
18 -0.436339   -0.31701884 -0.7060113
19  0.16666667 -0.51294726 -0.7060113
20  0.53934467  0.0        -0.7060113
 Faces:
 1  2  3  4  5.  1  6  7  8  2.  2  8  9 10  3.
 3 10 11 12  4.  4 12 13 14  5.  5 14 15  6  1.
20 19 18 17 16. 20 15 14 13 19. 19 13 12 11 18.
18 11 10  9 17. 17  9  8  7 16. 16  7  6 15 20.
```

Since this book is about programming aspects of graphics, a Java program to generate both above files (in a slightly different format) is listed below:

```
// IcoDode.java: Generating input files for
//    both an icosahedron and a dodecahedron.
//    Uses: Point3D (Section 3.9), Tria (Section 5.6).
import java.io.*;

public class IcoDode {
   public static void main(String[] args) throws IOException {
      new Both();
   }
}

class Both {
   Point3D[] icoV; Tria[] icoF;

   Both() throws IOException {
      outIcosahedron(); outDodecahedron();
   }

   void outIcosahedron() throws IOException {
      double zTop = 0.5 * Math.sqrt(5);
      icoV = new Point3D[13]; // icoV[1], ..., icoV[12]:
      icoV[1] = new Point3D(0, 0, zTop); // North pole
      double angle36 = Math.PI / 5, angle72 = 2 * angle36;
      for (int i = 2; i <= 6; i++) {
         double alpha = (i - 2) * angle72;
         icoV[i] = new Point3D(Math.cos(alpha),
            Math.sin(alpha), 0.5);
      }
      for (int i = 7; i <= 11; i++) {
         double alpha = angle36 + (i - 7) * angle72;
         icoV[i] = new Point3D(Math.cos(alpha),
            Math.sin(alpha), -0.5);
      }
```

```
      icoV[12] = new Point3D(0, 0, -zTop);
      icoF = new Tria[21]; // icoF[1], ..., icoF[20]
      icoF[1] = new Tria(1, 2, 3);
      icoF[2] = new Tria(1, 3, 4);
      icoF[3] = new Tria(1, 4, 5);
      icoF[4] = new Tria(1, 5, 6);
      icoF[5] = new Tria(1, 6, 2);
      icoF[6] = new Tria(2, 7, 3);
      icoF[7] = new Tria(3, 7, 8);
      icoF[8] = new Tria(3, 8, 4);
      icoF[9] = new Tria(4, 8, 9);
      icoF[10] = new Tria(4, 9, 5);
      icoF[11] = new Tria(5, 9, 10);
      icoF[12] = new Tria(5, 10, 6);
      icoF[13] = new Tria(6, 10, 11);
      icoF[14] = new Tria(6, 11, 2);
      icoF[15] = new Tria(2, 11, 7);
      icoF[16] = new Tria(12, 8, 7);
      icoF[17] = new Tria(12, 9, 8);
      icoF[18] = new Tria(12, 10, 9);
      icoF[19] = new Tria(12, 11, 10);
      icoF[20] = new Tria(12, 7, 11);
      FileWriter fwI = new FileWriter("icosa.dat");
      for (int i = 1; i <= 12; i++) {
         Point3D P = icoV[i];
         fwI.write(i + " " + P.x + " " + P.y + " " +
            P.z + "\r\n");
      }
      fwI.write("Faces:\r\n");
      for (int j = 1; j <= 20; j++) {
         Tria t = icoF[j];
         fwI.write(t.iA + " " + t.iB + " " + t.iC + ".\r\n");
      }
      fwI.close();
   }

   void outDodecahedron() throws IOException {
      FileWriter fwD = new FileWriter("dodeca.dat");
      for (int j = 1; j <= 20; j++)
         writeVertexInCenter(fwD, j);
      fwD.write("Faces:\r\n");
      // Horizontal, at the top:
      writeFace(fwD, 1, 2, 3, 4, 5);
      // Slightly facing upward:
      writeFace(fwD, 1, 6, 7, 8, 2);
```

```
      writeFace(fwD, 2, 8, 9, 10, 3);
      writeFace(fwD, 3, 10, 11, 12, 4);
      writeFace(fwD, 4, 12, 13, 14, 5);
      writeFace(fwD, 5, 14, 15, 6, 1);
      // Horizontal, at the bottom:
      writeFace(fwD, 20, 19, 18, 17, 16);
      // Slightly facing downward:
      writeFace(fwD, 20, 15, 14, 13, 19);
      writeFace(fwD, 19, 13, 12, 11, 18);
      writeFace(fwD, 18, 11, 10, 9, 17);
      writeFace(fwD, 17, 9, 8, 7, 16);
      writeFace(fwD, 16, 7, 6, 15, 20);
      fwD.close();
   }

   void writeVertexInCenter(FileWriter fwD, int j)
      throws IOException {
      Tria t = icoF[j];
      Point3D A = icoV[t.iA], B = icoV[t.iB], C = icoV[t.iC];
      float x = (float) ((A.x + B.x + C.x) / 3),
            y = (float) ((A.y + B.y + C.y) / 3),
            z = (float) ((A.z + B.z + C.z) / 3);
      fwD.write(j + " " + x + " " + y + " " + z + "\r\n");
   }

   void writeFace(FileWriter fwD, int a, int b, int c,
      int d, int e)
         throws IOException {
      fwD.write(a + " " + b + " " + c + " " + d + " "
         + e + ".\r\n");
   }
}
```

An icosahedron, as generated by this program, will be useful to generate (better) approximations of spheres, as we will see in the next section.

D.2. Sphere Representations

There are several ways of approximating a sphere by a polyhedron. A very popular one is the globe model with north and south poles at the top and the bottom, horizontal circles called *lines of latitude*, and circles called *lines of longitude* in vertical planes through the poles, as shown in Fig. D.4. A program to generate such a globe should preferably be based on a single integer n, indicating both the number

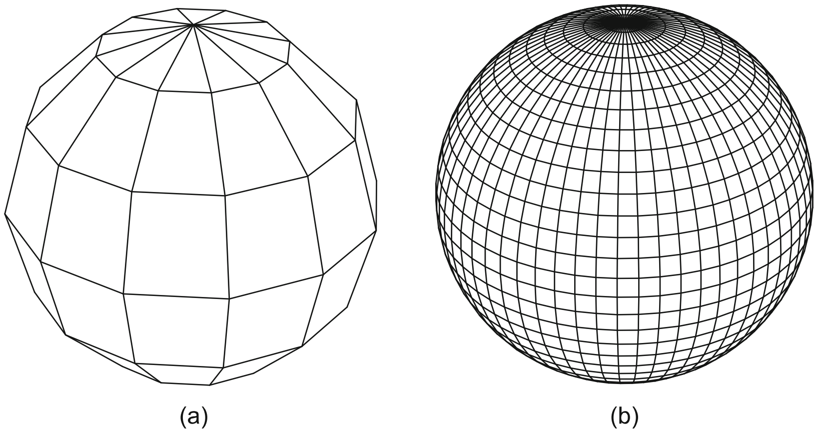


Fig. D.4 Approximations of a sphere; (**a**) $n = 6$; (**b**) $n = 30$

of horizontal slices and half the number of lines of longitude. In other words, angles of $180°/n$ play an essential role, both in horizontal and vertical planes. In Fig. D.4a, b, we have $n = 6$ and $n = 30$, respectively. Programming this is left as an exercise (see Exercise 5.7).

The above way of approximating a sphere has two drawbacks:

1. The faces are unequal in size and have different shapes: except for the triangles at the poles, each face is a trapezium, whose size depends on its distance from its nearest pole.
2. These spheres have an ‘anisotropic’ appearance: which may be undesirable. For example, a view from above looks much different from a view from the front. We may prefer all faces to look alike, so that the image should not significantly change if we turn the sphere a little.

We will therefore discuss different models of a sphere, in such a way that we cannot easily tell the poles and the other vertices apart.

Spheres Based on an Icosahedron

As Fig. D.3 illustrates, an icosahedron and a dodecahedron are very poor approximations of a sphere. However, we can use the former as a basis to generate a much better one. The idea is to divide each triangular face of an icosahedron into four triangles, as shown in Fig. D.5, and to construct points on the sphere to which all vertices of the icosahedron belong. For the icosahedron we have constructed, the center of this sphere is O and its radius is $\frac{1}{2}\sqrt{5}$.

With O as the center of the icosahedron and D, E and F as the midpoints of three triangle edges BC, CA and AB, we have to extend the lines OD, OE and OF, such that the new vertices D, E and F (like A, B and C) lie on the sphere just mentioned.

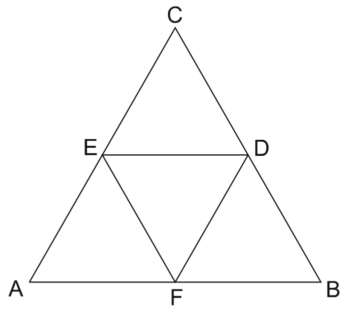


Fig. D.5 Dividing a triangle into four smaller ones

Actually, we may as well change the lengths of all six lines OA, OB, . . ., OF, such that the new points A, B, . . ., F lie on a sphere with center O and radius 1. Doing this for all 20 faces of an icosahedron, we obtain a polyhedron that has $20 \times 4 = 80$ triangles. Since each of the 30 original edges gives a new vertex (its midpoint), and there were already 12 vertices, this new polyhedron has $12 + 30 = 42$ vertices. It follows from Eq. (D.1) that it has $80 + 42–2 = 120$ edges. Note that the four triangles of Fig. D.5 are equilateral (that is, they have three edges of the same length), but that this is the case only as long as the six points A, B, . . ., F lie in the same plane. Consequently, the 80 faces of our new polyhedron are *not* equilateral triangles.

Instead of writing a program just to generate this polyhedron, we will make the program much more general. It will accept the names of an input and an output file. The input file can be any file (in our 3D format) for a convex polyhedron that has only triangular faces and whose center is the origin O of the coordinate system (provided the vertex numbers 1, 2, 3, . . . are used and appear in that order in the first part of the file). In the output file we obtain a polyhedron that has four times as many (triangular) faces as the given one, and whose vertices lie on a sphere with center O and radius 1. This will enable us to use the program several times, starting with the file *icosa.dat*, generated by program *IcoDode.java*, and each next time using the previous output file as input. Using names of the form *sph**x**.dat*, where ***x*** is the number of faces, we can proceed as follows:

$$icosa.dat \Rightarrow sph80.dat \Rightarrow sph320.dat \Rightarrow sph1280.dat \Rightarrow \quad \ldots$$

Figure D.6 shows the results of applying program *HLines.java* to these four files, of which the last three have been generated by the program we are discussing.

Program *SphTria.java* takes the names of its input and output files as program arguments. After having generated the file *icosa.dat* by using program *IcoDode.java* of Sect. D.1, we obtain the files for the other three above sphere approximations as follows:

```
java SphTria icosa.dat sph80.dat
java SphTria sph80.dat sph320.dat
java SphTria sph320.dat sph1280.dat
```

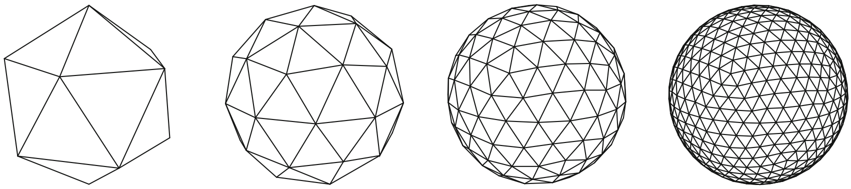


Fig. D.6 Icosahedron and three other polyhedra derived from it by program SphTria.java: (**a**) 20 faces (see Sect. D.1); (**b**) 80 faces; (**c**) 320 faces; (**d**) 1280 faces

Like some other programs in this book, *SphTria.java*, listed below, uses the classes *Point*3*D*, *Tria* and *Input*, discussed in Sects. 3.9 and 5.6:

```
// SphTria.java: Generating a 3D object file for a sphere
//     approximation consisting of triangles. In the output file there
//     are four times as many triangles as in the input file. Suitable
//     input files are icosa.dat, produced by program IcoDode.java, and
//     the output files produced by this program (SphTria.java) itself!
//     To run this program, enter, for example,
//        java SphTria icosa.dat sph80.dat.
//     Uses: Tria, Input (Section 5.6), Point3D (Section 3.9).

import java.io.*;
import java.util.*;

public class SphTria {
   public static void main(String[] args) throws IOException {
      if (args.length < 2) {
         System.out.println(
               "Command:\n" + "java SphTria InputFile OutputFile");
      } else
         new SphTriaObj(args[0], args[1]);
   }
}

class SphTriaObj {
   Vector v = new Vector(); // Vertices
   Vector<Tria> t = new Vector<Tria>(); // Triangular faces
   int nV, codeRadix;
   Hashtable<Integer, Integer> ht = new Hashtable<Integer, Integer>();
   String inputFile, outputFile;

   SphTriaObj(String inputFile, String outputFile) throws IOException {
      this.inputFile = inputFile;
      this.outputFile = outputFile;
```

```
      readFile();
      computeMidpoints();
      toUnitCircle();
      writeFile();
   }

   void readFile()/* throws IOException*/ {
      Input inp = new Input(inputFile);
      if (inp.fails()) error();
      v.addElement(new Integer(0)); // Start at position 1
      for (;;) {
         int nr = inp.readInt();
         if (inp.fails()) break;
         nV = nr;
         float x = inp.readFloat(), y = inp.readFloat(),
              z = inp.readFloat();
         v.addElement(new Point3D(x, y, z));
      }
      inp.clear();
      codeRadix = nV + 1;
      inp.clear();
      while (inp.readChar() != '\n' && !inp.fails())
         ;
      // Rest of line 'Faces:' has now been skipped.
      for (;;) {
         int a = inp.readInt(), b = inp.readInt(), c = inp.readInt();
         if (inp.fails()) break;
         t.addElement(new Tria(a, b, c));
         inp.readChar(); // Skip '.'
      }
      inp.clear();
   }

   void error() {
      System.out.println("Problem with file input file " + inputFile);
      System.exit(1);
   }

   void computeMidpoints() {
      for (int j = 0; j < t.size(); j++) {
         Tria tr = (Tria) t.elementAt(j);
         int a = tr.iA, b = tr.iB, c = tr.iC;
         addMidpoint(a, b); addMidpoint(b, c); addMidpoint(c, a);
      }
   }
```

```
  void addMidpoint(int p, int q) {
     if (p < q) {
        ht.put(new Integer(codeRadix * p + q), new Integer(++nV));
        Point3D P = (Point3D) v.elementAt(p), Q = (Point3D) v
              .elementAt(q);
        v.addElement(new Point3D( // at position nV
             (P.x + Q.x) / 2, (P.y + Q.y) / 2, (P.z + Q.z) / 2));
     }
  }

  int getMidpoint(int p, int q) {
    int key = p < q ? (codeRadix * p + q) : (codeRadix * q + p);
    Integer iObj = (Integer) ht.get(new Integer(key));
    return iObj.intValue();
  }

  void toUnitCircle() {
     for (int i = 1; i <= nV; i++) // nV = v.size() - 1
     {
        Point3D P = (Point3D) v.elementAt(i);
        float r = (float) Math.sqrt(P.x * P.x + P.y * P.y + P.z * P.z);
        P.x /= r; P.y /= r; P.z /= r;
     }
  }

  void writeFile() throws IOException {
     FileWriter fw = new FileWriter(outputFile);
     for (int i = 1; i < v.size(); i++) {
        Point3D P = (Point3D) v.elementAt(i);
        fw.write(i + " " + P.x + " " + P.y + " " + P.z + "\r\n");
     }
     fw.write("Faces\r\n");
     for (int j = 0; j < t.size(); j++) {
        Tria tr = (Tria) t.elementAt(j);
        int a = tr.iA, b = tr.iB, c = tr.iC;
        int mab = getMidpoint(a, b), mbc = getMidpoint(b, c),
            mca = getMidpoint(
              c, a);
        fw.write(a + " " + mab + " " + mca + ".\r\n");
        fw.write(b + " " + mbc + " " + mab + ".\r\n");
        fw.write(c + " " + mca + " " + mbc + ".\r\n");
        fw.write(mab + " " + mbc + " " + mca + ".\r\n");
     }
     fw.close();
  }
}
```

Each edge of a triangle of the given polyhedron is also an edge of a neighboring triangle. Since for each triangle its vertices are specified in counter-clockwise order, the ordered vertex number pair *p*, *q* representing an edge of one of these two triangles will appear as *q*, *p* for the other triangle. To avoid adding the midpoint of this edge twice, we accept the pair *p*, *q* only if *p* is less than *q*, as the test at the beginning of the method *addMidpoint* shows. Note that we process all given triangles twice, first, in *computeMidpoints*, to add the midpoint of every edge (exactly once) to the list of vertices *v*, and later, in *writeFaces*, to construct the new, smaller triangles. In the first of these two actions, we associate a new midpoint, say, vertex number *m*, with a certain edge *p*, *q*. This must be stored in such a way, that later, in *writeFaces*, when we encounter this edge *p*, *q* again, we can use this pair to find this same vertex number *m*. We can do this efficiently and conveniently by using a *hash table*, writing.

```
Hashtable ht = new Hashtable();
```

The pair *p*, *q*, or rather, a large integer *codeRadix* * *p* + *q*, where *codeRadix* is a large constant and $p < q$, is used as a key. By storing the pair (*key*, *m*) in the hash table *ht*., we can later retrieve the midpoint vertex number *m* using the same key. This explains both the statement

```
ht.put(new Integer(codeRadix * p + q), new Integer(++nV));
```

in *addMidpoint* (called in *computeMidpoints*) and

```
Integer iObj = (Integer)ht.get(new Integer(key));
```

in *getMidpoint* (called in *writeFile*).

The sphere that program *SphTria.java* produces obviously depends upon its input file. The file *tetra.dat* of Sect. D.1, like *iscosa.dat*, describes a polyhedron consisting of equilateral triangles and has O as its center. Note that the latter is not the case with the file *tetra*1*.dat*, obtained by the program *Rota3DTetra.java* because this program rotates the tetrahedron about the edge 1-2. Another polyhedron satisfying the conditions just mentioned is the octahedron, also discussed in Sect. D.1. This gives a better, though rather peculiar, result. If we start with the file *octa.dat*, describing the octahedron shown in Fig. D.7a, we obtain

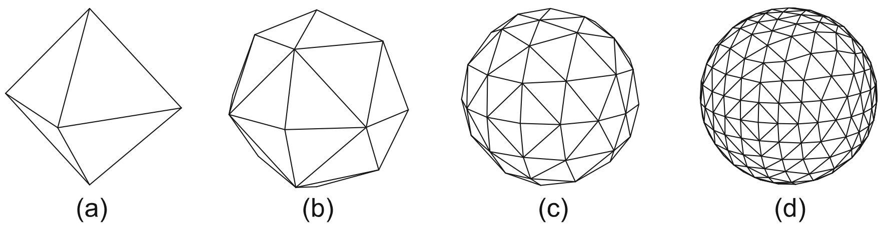


Fig. D.7 Octahedron and three other polyhedra derived from it by program *SphTria.java*: (**a**) 8 faces (see Sect. D.1); (**b**) 32 faces; (**c**) 128 faces; (**d**) 512 faces

$$octa.dat \Rightarrow sph32.dat \Rightarrow sph128.dat \Rightarrow sph512.dat$$

Note that Fig. D.7b–d show squares or square-like patterns around the original vertices of the octahedron. We can see this effect here only for the nearest vertex of Fig. D.7a, but it applies to all six vertices of the octahedron.

D.3. A Torus

We will now discuss a program to construct a *torus*, as shown in Fig. D.8. The input data of this program will consist of three program arguments:

- n, the number of small vertical circles; on each of these, n points are used as vertices to approximate the torus;
- R, the radius of a large horizontal circle containing the centers of the n smaller circles;
- The name of the output file.

All small circles will have the same radius, $r = 1$. You can see the large horizontal circle (with radius R) and one of the small vertical ones in Fig. D.9. Since the smaller circles must not intersect each other, we require that

$$R \geq r(= 1)$$

A parametric representation of the large circle is

$$\begin{aligned} x &= R\cos\alpha \\ y &= R\sin\alpha \\ z &= 0 \end{aligned}$$

If we take $\alpha = 0$ in these equations, we obtain the center of the small vertical circle, shown in Fig. D.9, which has the following parametric representation (where $r = 1$):

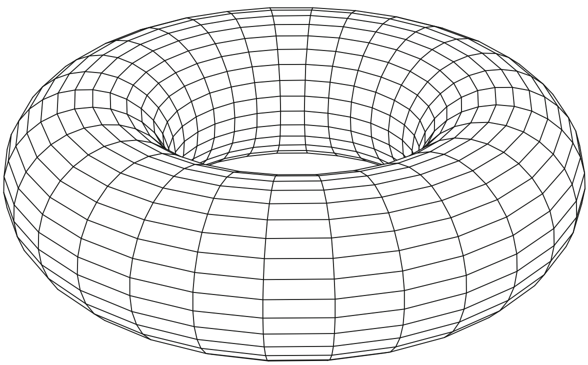


Fig. D.8 Torus ($n = 30$ and $R = 2.5$)

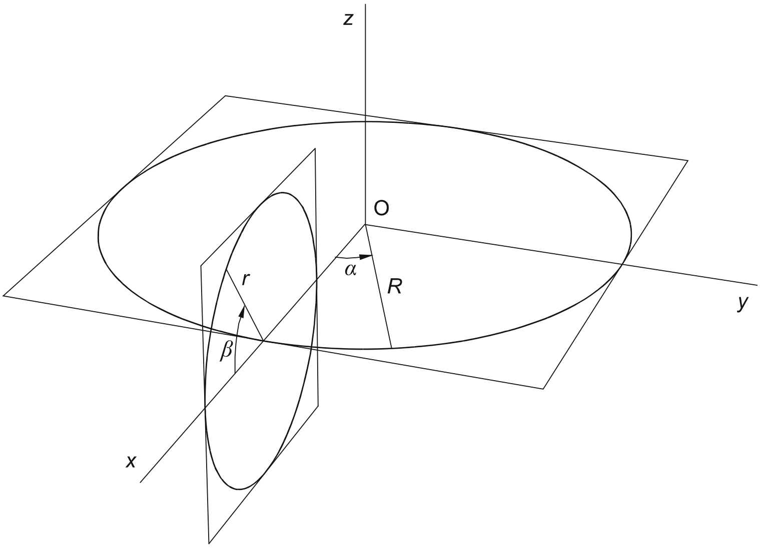


Fig. D.9 Basic circles of a torus

$$x = R + r \cos \beta$$
$$y = 0$$
$$z = r \sin \beta$$

This small circle belongs to $i = 0$. By rotating it about the z-axis through angles $\alpha = i\delta$, where $i = 1, \ldots, n - 1$ and $\delta = 2\pi/n$, we obtain the remaining $n - 1$ small circles. As for the vertex numbers of the torus, we select n points on the first small circle (corresponding to $i = 0$) and assign the integers 1, 2, ..., n to them: the point obtained by using parameter $\beta = j\delta$ is assigned vertex number $j + 1$ ($j = 0, 1, ..., n - 1$). The next n vertices, numbered $n + 1, n + 2, ..., 2n$, lie on the neighboring circle, corresponding to $i = 1$, and so on. In general, we use the n^2 vertex numbers $i \cdot n + j + 1$ ($i = 0, 1, ..., n - 1$; $j = 0, 1, ..., n - 1$).

We will now rotate the small vertical circle drawn in Fig. D.7 about the z-axis through the angle $\alpha = i\delta$. As follows from Sect. 3.2, this rotation can be written

$$[x' \quad y'] = [x \quad y] \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

In our case the basic small circle lies in the xz-plane, so that $y = 0$, which reduces this matrix product to

$$x' = x \cos \alpha$$
$$y' = x \sin \alpha$$

(which we could also have derived directly from Fig. D.9, without using the above matrix multiplication).

As you can see in the following program, we use most of the above formulas in the inner loop, in which the variable x denotes the x-coordinate of a point of the small circle in the xz-plane (see Fig. D.9), while $x1$ denotes the x-coordinate of that point after rotation through α about the z-axis:

```
// Torus.java: Generating a data file for a torus. R is the radius of
//     a large horizontal circle, on which n equidistant points will be
//     the centers of small vertical circles with radius 1. The values
//     of n and R as well as the output file name are to be supplied as
//     program arguments.
import java.io.*;

public class Torus {
   public static void main(String[] args) throws IOException {
      if (args.length != 3) {
         System.out.println(
"Supply n (> 2), R (>= 1) " + "and a filename as program arguments.\n");
         System.exit(1);
      }
      int n = 0;
      double R = 0;
      try {
         n = Integer.valueOf(args[0]).intValue();
         R = Double.valueOf(args[1]).doubleValue();
         if (n <= 2 || R < 1)
            throw new NumberFormatException();
      } catch (NumberFormatException e) {
         System.out.println("n must be an integer > 2");
         System.out.println("R must be a real number >= 1");
         System.exit(1);
      }
      new TorusObj(n, R, args[2]);
   }
}

class TorusObj {
   TorusObj(int n, double R, String fileName) throws IOException {
      FileWriter fw = new FileWriter(fileName);
      double delta = 2 * Math.PI / n;
      for (int i = 0; i < n; i++) {
         double alpha = i * delta,
                cosa = Math.cos(alpha), sina = Math.sin(alpha);
         for (int j = 0; j < n; j++) {
            double beta = j * delta, x = R + Math.cos(beta);
```

```
            float x1 = (float) (cosa * x),
                  y1 = (float) (sina * x),
                  z1 = (float) Math.sin(beta);
            fw.write(
            (i * n + j + 1) + " " + x1 + " " + y1 + " " + z1 + "\r\n");
         }
      }
      fw.write("Faces:\r\n");
      for (int i = 0; i < n; i++) {
         for (int j = 0; j < n; j++) {
            int i1 = (i + 1) % n, j1 = (j + 1) % n,
                a = i * n + j + 1, b = i1 * n + j + 1,
                c = i1 * n + j1 + 1, d = i * n + j1 + 1;
            fw.write(a + " " + b + " " + c + " " + d + ".\r\n");
         }
      }
      fw.close();
   }
}
```

D.4. Beams in a Spiral

Our next example is a spiral as shown in Fig. D.11a–c. It is built from horizontal beams with length $2a$ ($a \geq 0.5$), width 1 and height 1. The bottom of the lowest beam lies in the xy-plane; this beam is parallel to the y-axis and its maximum x-coordinate is equal to a (see Fig. D.10). Each next beam can be obtained by lifting the previous one a distance 1, and by rotating it about the z-axis though a given

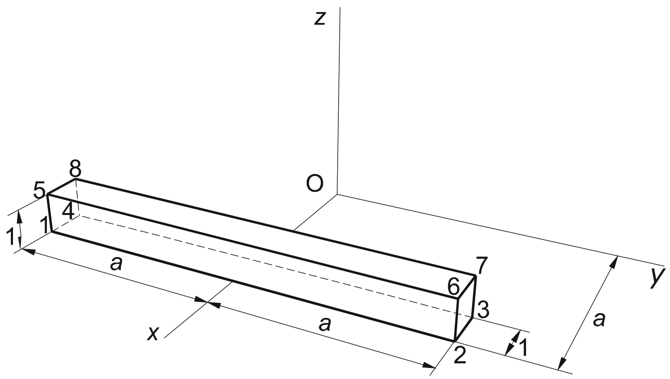


Fig. D.10 Vertex numbers of first beam

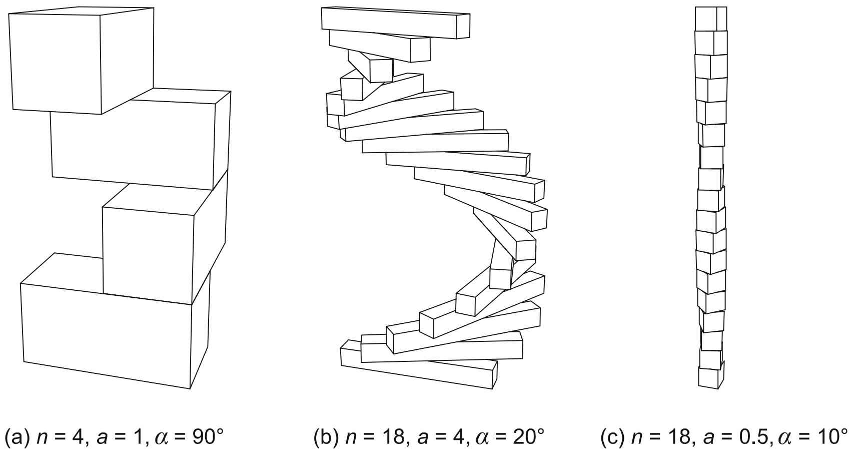


Fig. D.11 Spirals of beams

angle α. There are n beams. The integer n, the two real numbers a and α (the latter in degrees) and the output file name, in that order, are supplied as program arguments.

The beams in Fig. D.11a–c have lengths ($2a =$) 2, 8 and 1, respectively, which implies that the beams are actually cubes in Fig. D.11c.

The data files for program *HLines.java* to produce the three spiral images of Fig. D.11 were generated by running program *Beams.java* three times as follows:

```
java Beams 4 1 90 FigD_11a.dat
java Beams 18 4 20 FigD_11b.dat
java Beams 18 0.5 10 FigD_11c.dat
```

Although we could have used the class *Rota3D*, discussed in Sect. 3.9, the rotation about the z-axis is as simple as a 2D rotation about O (see Sect. 3.2), so that we may as well program it directly, as is done in the nested for-loop in program *Beams.java*, listed below:

```
// Beams.java: Generating input files for a spiral of beams. The
// values of n, a and alpha (in degrees) as well as the output
// file name are to be supplied as program arguments.
// Uses: Point3D (Section 3.9).
import java.io.*;

public class Beams {
   public static void main(String[] args) throws IOException {
      if (args.length != 4) {
```

```
            System.out.println(
                    "Supply n (> 0), a (>= 0.5), alpha (in degrees)\n" +
                    "and a filename as program arguments.\n");
            System.exit(1);
        }
        int n = 0;
        double a = 0, alphaDeg = 0;
        try {
            n = Integer.valueOf(args[0]).intValue();
            a = Double.valueOf(args[1]).doubleValue();
            alphaDeg = Double.valueOf(args[2]).doubleValue();
            if (n <= 0 || a < 0.5)
                throw new NumberFormatException();
            } catch (NumberFormatException e) {
              System.out.println("n must be an integer > 0");
              System.out.println("a must be a real number >= 0.5");
              System.out.println("alpha must be a real number");
              System.exit(1);
            }
            new BeamsObj(n, a, alphaDeg * Math.PI / 180, args[3]);
        }
}

class BeamsObj {
   FileWriter fw;

   BeamsObj(int n, double a, double alpha, String fileName)
         throws IOException {
      fw = new FileWriter(fileName);
      Point3D[] P = new Point3D[9];
      double b = a - 1;
      P[1] = new Point3D(a, -a, 0);
      P[2] = new Point3D(a, a, 0);
      P[3] = new Point3D(b, a, 0);
      P[4] = new Point3D(b, -a, 0);
      P[5] = new Point3D(a, -a, 1);
      P[6] = new Point3D(a, a, 1);
      P[7] = new Point3D(b, a, 1);
      P[8] = new Point3D(b, -a, 1);
      for (int k = 0; k < n; k++) { // Beam k:
         double phi = k * alpha,
                cosPhi = Math.cos(phi), sinPhi = Math.sin(phi);
         int m = 8 * k;
```

```
            for (int i = 1; i <= 8; i++) {
               double x = P[i].x, y = P[i].y;
                float x1 = (float) (x * cosPhi - y * sinPhi),
                     y1 = (float) (x * sinPhi + y * cosPhi),
                     z1 = (float) (P[i].z + k);
                fw.write((m + i) + " " + x1 + " " + y1 + " " + z1 + "\r\n");
            }
        }
        fw.write("Faces:\r\n");
        for (int k = 0; k < n; k++) { // Beam k again:
           int m = 8 * k;
           face(m, 1, 2, 6, 5);
           face(m, 4, 8, 7, 3);
           face(m, 5, 6, 7, 8);
           face(m, 1, 4, 3, 2);
           face(m, 2, 3, 7, 6);
           face(m, 1, 5, 8, 4);
        }
        fw.close();
}

 void face(int m, int a, int b, int c, int d) throws IOException {
    a += m;
    b += m;
    c += m;
    d += m;
    fw.write(a + " " + b + " " + c + " " + d + ".\r\n");
 }
}
```

D.5. Functions of Two Variables

Although the programs *Painter.java*, *ZBuf.java* and *HLines.java* were primarily intended to represent solid objects, they can also be used for other purposes, such as displaying surfaces that correspond to functions of the form

$$z = f(x, y)$$

For example, Fig. D.12 shows such a surface for the function

$$z = f(x, y) = \frac{10\cos\sqrt{x^2 + y^2}}{2 + \sqrt{x^2 + y^2}} \tag{D.2}$$

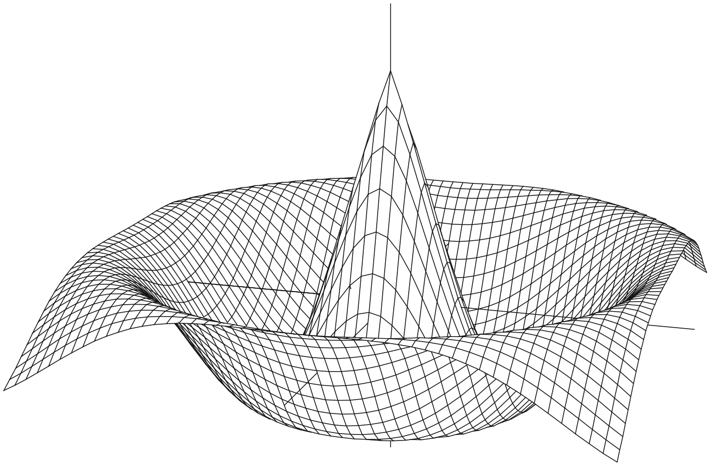


Fig. D.12 Surface of a function $z = f(x, y)$ as specified by (D.2 and D.3)

This line drawing was made by exporting an HP-GL file, as discussed in Appendix C. As usual, the x-axis points towards us, the y-axis points to the right and the z-axis points upwards. Note that each of the three coordinate axes is partly visible and partly hidden. The lines on the surface connect points (x, y, z), where

$$-6 \leq x \leq 6 \text{ and } -6 \leq y \leq 6, \text{ with step sizes } \Delta x = \Delta y = 0.25 \qquad \text{(D.3)}$$

Instead of dealing only with this particular function, we will discuss a very general program, based on an *expression evaluator*. Using a graphical user interface, we will enable the user to enter both an expression for the function in question and the intervals and step sizes for x and y. Such expressions are similar to what we write in our programs, although the set of available operators and standard functions is very limited:

1. An *expression* consists of one or more terms, separated by + and –.
2. A *term* consists of one or more factors, separated by * and /.
3. A *factor* can be
   (a) a real number (such as 12.3, 4 or -25; number representations such as 1e7 are not allowed),
   (b) a variable x or a variable y (and no other variables),
   (c) an expression, as defined in point 1, written between parentheses, such as the one occurring in 3 * $(x + y)$,
   (d) a function call of one of the following three forms (and no others):

   sin(*expression*)
   cos(*expression*)
   pow(*expression*, *expression*).

The last three standard functions are the same as those in the *Math* class of Java. For example, we write *pow*(x, 0.5) for $x^{0.5}$. There is no special *sqrt* function. As usual, blank spaces are allowed in expressions.

For example, the function given by Eq. (D.2) can be written as the following expression:

```
10 * cos(pow(x*x + y*y, 0.5))/(2 + pow(x*x + y*y, 0.5))
```

As for the intervals and step sizes for x and y, we write, for example,

```
-6 (0.25) 6
```

indicating that the variable in question ranges from -6 to 6, with step size 0.25. The dialog box to be used to enter expressions, intervals and step sizes is shown in Fig. D.13.

There is also a text field in which the user has to enter the name of the output file. After filling in all text fields, the user can click the button *Write file* to generate the desired file. Since this does not terminate the program, there is also an *Exit* button. Although the program *Func.java* produces only a file, such as *cospow.dat* in this example, we can quickly see the 3D surface in question by executing the two programs *Func.java* and *HLines.java* at the same time. After pressing the button *Write file*, in the former program, we use *File* | *Open* in the latter to see the result. If we want to change any data, such as shown in Fig. D.13, we go back to *Func.java*, which is still running; after doing this, we click the *Write file* button and switch to *HLines.java* again, and so on.

If the expression entered for $z = f(x, y)$ cannot be interpreted because it is syntactically incorrect, the computer gives an audible signal. Such a beep also occurs if an interval or a step size is incorrect; this happens, for example, if *xMin* is greater than *xMax* – *deltaX*. It follows from this that error handling is not the strongest point of the program we will discuss, and neither is its speed. However, although the given expression is not converted into more efficient intermediate code (such as postfix), it is much faster than the hidden-line program that follows;

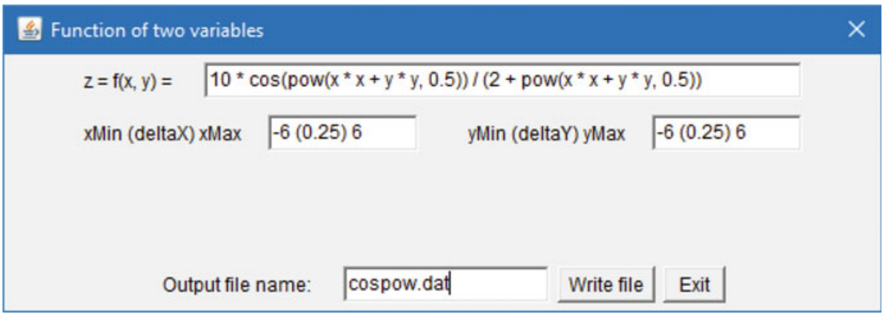


Fig. D.13 Dialog box for program *Func.java* improvements with regard to error handling and efficiency were deliberately omitted because many readers will find the program already complex enough in its current form. After having a look at the program listed below, we will discuss some of its internal aspects.

```
// Func.java: A function of two variables x and y.
import java.awt.*;
import java.awt.event.*;
import java.io.*;

public class Func extends Frame {
   public static void main(String[] args) {new Func();}
   Func() {new FuncDialog(this);}
}

class FuncDialog extends Dialog {
   TextField tfFun = new TextField(50),
      tfX = new TextField(10), tfY = new TextField(10),
      tfFileName = new TextField(15);
   Button buttonWriteFile = new Button("Write file"),
      buttonExit = new Button(" Exit ");

   FuncDialog(Frame fr) {
      super(fr, "Function of two variables", true);

      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {
            dispose(); System.exit(0);
         }
      });

      Panel p1 = new Panel(), p2 = new Panel(),
         p3 = new Panel();
      p1.add(new Label("z = f(x, y) = ")); p1.add(tfFun);
      p2.add(new Label("xMin (deltaX) xMax")); p2.add(tfX);
      p2.add(new Label("        yMin (deltaY) yMax"));
      p2.add(tfY);
      p3.add(new Label("Output file name: "));
      p3.add(tfFileName);
      p3.add(buttonWriteFile); p3.add(buttonExit);
      setLayout(new BorderLayout());
      add("North", p1); add("Center", p2); add("South", p3);

      buttonWriteFile.addActionListener(
         new ActionListener() {
         public void actionPerformed(ActionEvent ae) {
```

```
            float xa = 0, dx = 0, xb = 0,
                  ya = 0, dy = 0, yb = 0;
            String sX = tfX.getText();

            xa = (new xyExpression(sX)).factor();
            sX = sX.substring(sX.indexOf('(') + 1);
            dx = (new xyExpression(sX)).factor();
            sX = sX.substring(sX.indexOf(')') + 1);
            xb = (new xyExpression(sX)).factor();

            String sY = tfY.getText();
            ya = (new xyExpression(sY)).factor();
            sY = sY.substring(sY.indexOf('(') + 1);
            dy = (new xyExpression(sY)).factor();
            sY = sY.substring(sY.indexOf(')') + 1);
            yb = (new xyExpression(sY)).factor();

            if (xa + dx > xb || dx <= 0 ||
               ya + dy > yb || dy <= 0) {
               Toolkit.getDefaultToolkit().beep();
               return;
            }
            String s = tfFun.getText(),
               fileName = tfFileName.getText();

            xyExpression xyE = new xyExpression(s);
            try {
               xyE.generate(xa, dx, xb, ya, dy, yb,
                  fileName);
            } catch (IOException ioe) {
            }
         }
      });
      buttonExit.addActionListener(new ActionListener() {
         public void actionPerformed(ActionEvent ae) {
            dispose(); System.exit(0);
         }
      });

      Dimension dim = getToolkit().getScreenSize();
      setSize(6 * dim.width / 10, dim.height / 4);
      setLocation(dim.width / 5, dim.height / 2);
      setVisible(true);
   }
}
```

```
class xyExpression {
   String buf;
   float x, y, lastNum;
   char lastChar;
   int pos;
   boolean OK;

   xyExpression(String s) {buf = s; OK = true;}

   void generate(float xa, float dx, float xb,
                 float ya, float dy, float yb,
                 String fileName) throws IOException {
      FileWriter fw = new FileWriter(fileName);
      int nx = Math.round((xb - xa) / dx),
          ny = Math.round((yb - ya) / dy), nr = 0;
      float za = 1e30F, zb = -1e30F;
      outer:
         for (int j = 0; j <= ny; j++) {
         float y = ya + j * dy;
         for (int i = 0; i <= nx; i++) {
            float x = xa + i * dx;
            nr = j * (nx + 1) + i + 1;
            float z = eval(x, y);
            if (!OK) {
               Toolkit.getDefaultToolkit().beep();
               break outer;
            }
            if (z < za) za = z;
            if (z > zb) zb = z;
            fw.write(nr + " " + x + " " + y + " "
               + z + "\r\n");
         }
      }
      // x, y and z axes:
      float dz = (zb - za) / 10,
         xa1 = Math.min(xa - 2 * dx, 0),
         xb1 = Math.max(xb + 2 * dx, 0),
         ya1 = Math.min(ya - 2 * dy, 0),
         yb1 = Math.max(yb + 2 * dy, 0),
         za1 = Math.min(za - 2 * dz, 0),
         zb1 = Math.max(zb + 2 * dz, 0);
      fw.write(++nr + " " + xa1 + " 0 0\r\n");
      fw.write(++nr + " " + xb1 + " 0 0\r\n");
      fw.write(++nr + " 0 " + ya1 + " 0\r\n");
      fw.write(++nr + " 0 " + yb1 + " 0\r\n");
      fw.write(++nr + " 0 0 " + za1 + "\r\n");
```

```
      fw.write(++nr + " 0 0 " + zb1 + "\r\n");
      fw.write("Faces:\r\n");
      for (int i = 0; i < nx; i++) {
         for (int j = 0; j < ny; j++) {
            int k = j * (nx + 1) + i + 1,
                m = k + nx + 1, k1 = k + 1, m1 = m + 1;
            fw.write(k + " " + -m1 + " " + k1 + ".\r\n");
            fw.write(k1 + " " + m1 + " " + -k + ".\r\n");
            fw.write(k + " " + -m1 + " " + m + ".\r\n");
            fw.write(m + " " + m1 + " " + -k + ".\r\n");
         }
      }
      int k = (nx + 1) * (ny + 1);
      fw.write(++k + " " + ++k + ".\r\n"); // x-axis
      fw.write(++k + " " + ++k + ".\r\n"); // y-axis
      fw.write(++k + " " + ++k + ".\r\n"); // z-axis
      fw.close();
      System.out.println("Ready!");
   }

   boolean readChar() {
      char ch;
      do {
         if (pos == buf.length()) return false;
         ch = buf.charAt(pos++);
      } while (ch == ' ');
      lastChar = ch; return true;
   }

   boolean nextIs(char ch) {
      char ch0 = lastChar;
      if (readChar()) {
         if (ch == lastChar) return true;
         pos--;
      }
      lastChar = ch0; return false;
   }

   float eval(float x, float y) {
      this.x = x; this.y = y; pos = 0; OK = true;
      return expression();
   }
```

```
float expression() {
   float x = term();
   for (;;) {
      if (nextIs('+')) x += term(); else
      if (nextIs('-')) x -= term(); else break;
   }
   return x;
}

float term() {
   float x = factor();
   for (;;) {
      if (nextIs('*')) x *= factor(); else
      if (nextIs('/')) x /= factor(); else break;
   }
   return x;
}

float factor() {
   float v = 0;
   if (!readChar()) return 0;
   if (lastChar == 'x') return x;
   if (lastChar == 'y') return y;
   if (lastChar == '(') {
      v = expression();
      if (!nextIs(')')) {OK = false; return 0;}
      return v;
   }
   char ch = lastChar;
   if (ch == 'c'    // cos(expression)
    || ch == 's'    // sin(expression)
    || ch == 'p') { // pow(expression, expression)
      while ((OK = readChar()) && lastChar != '(')
         ;
      if (!OK) return 0;
      float arg = expression();
      if (ch == 'p') {
         if (!nextIs(',')) {OK = false; return 0;}
         double exponent = expression();
         v = (float) Math.pow(arg, exponent);
      } else
         v = (float)
            (ch == 'c' ? Math.cos(arg) : Math.sin(arg));
      if (!nextIs(')')) {OK = false; return 0;}
      return v;
   }
```

```
      pos--;
      if (number()) return lastNum;
      OK = false; return 0;
   }

   boolean number() {
      float x = 0;
      int nDec = -1;
      boolean neg = false;
      do {
         if (!readChar()) return false;
         if (lastChar == '-') {
            neg = true;
            if (!readChar()) return false;
            break;
         }
      } while (Character.isWhitespace(lastChar));
      if (lastChar == '.') {
         if (!readChar()) return false;
         nDec = 0;
      }
      if (!Character.isDigit(lastChar)) {
         OK = false; return false;
      }
      for (;;) {
         if (lastChar == '.' && nDec < 0) nDec = 0; else
         if (Character.isDigit(lastChar)) {
            x = 10 * x + (lastChar - '0');
            if (nDec >= 0) nDec++;
         }
         else {
            pos--; break;
         }
         if (!readChar()) break;
      }
      while (nDec > 0) {x *= 0.1; nDec--;}
      lastNum = (neg ? -x : x);
      return true;
   }
}
```

The given expression is evaluated when the first part of the output file is generated. In the class *xyExpression* we find the method *generate*, in which, five lines after the label *outer*, the following call to *eval* occurs:

```
float z = eval(x, y);
```

This method *eval* calls the method *expression*, whose task is to scan an expression, stored in the array *buf*, and to return its value. According to our above definition of *expression*, defining an expression as a sequence of terms separated by + and – operators, the method *expression* calls the method *term*, which scans a term and returns its value, and so on. This way of *parsing*, that is, analyzing expressions that satisfy a given *grammar*, is referred to as *recursive descent*. This adjective *recursive* will be clear if we note that our above syntactic definition of *expression* and the corresponding method are recursive. An *expression* can contain a *factor* that again contains an *expression*. Accordingly, in the program, *expression* calls *term*, which calls *factor*, which may again call *expression*. We immediately evaluate the syntactic entities we are dealing with, or, as we normally say, we *interpret* the source code. Instead, we might have generated intermediate code. More information about recursive descent parsing can be found in *Algorithms and Data Structures in C++* (see Bibliography).

In the program, parsing is done by using two simple methods, *readChar* and *nextIs*, as well as the variable *lastChar*, all belonging to the class *xyExpression*:

`boolean readChar()`
: This method scans the next character, if possible, places this in the variable *lastChar* and returns *true*. If this is not possible because the end of the expression is encountered, it returns *false*. This methods skips any blank spaces.

`boolean nextIs(char ch)`
: This method tests if the next character to be scanned is equal to the argument *ch*. If so, it scans this character and returns *true*; if not, it leaves the scan position unaltered and returns *false*.

After this discussion of parsing (which is unusual in a book on computer graphics) let us now turn to the graphics aspects of this program. It goes without saying that the vertices of a polygon, as specified in our 3D data files after the word *Faces*, should lie in the same plane. This is obviously the case for rectangles (or squares) such as 7-8-12-11 in Fig. D.14a if we regard these as lying in the plane $z = 0$.

However, the corresponding four points on the surface may or may not lie in the same plane. We therefore prefer triangles to rectangles in this case. Remember, the three points of a triangle always lie in the same plane. Instead of the rectangle

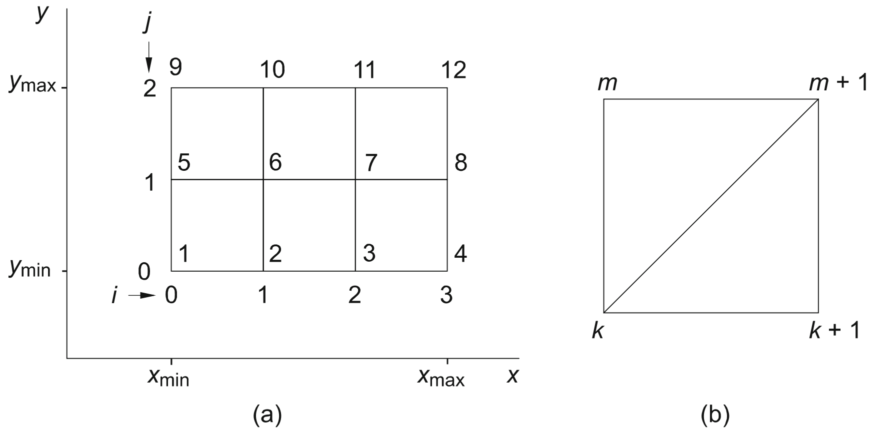


Fig. D.14 Two triangles with an invisible common edge forming a rectangle

7-8-12-11, we can specify the two triangles 7-8-12 and 7-12-11. However, we must pay attention to these two aspects:

1. Diagonals, such as 7-12, must not be drawn; we solve this problem by using minus signs, as discussed in Sect. 5.5.
2. Each triangle has two sides; since we do not know in advance which will be visible, we have to specify both. Recall that we have discussed this subject of 'individual faces' in Sect. 5.5.

According to this second point, the vertices of each triangle in Fig. D.14b will occur twice in the data file: clockwise and counter-clockwise. Using minus signs, as discussed in the first point, we use four lines in the data file for the 'rectangle' k, $k + 1$, $m + 1$, m of Fig. D.14b, as shown below:

k	$-(m + 1)$	$k + 1.$	(Lower-right, clockwise)
$k + 1$	$m + 1$	$-k.$	(Lower-right, counter-clockwise)
k	$-(m + 1)$	$m.$	(Upper-left, counter-clockwise)
m	$m + 1$	$-k.$	(Upper-left, clockwise)

You can find the code that writes these four triangles to a file in the method *generate*.

Painting Faces Instead of Drawing Lines

So far, in this Appendix, we used line drawings to display the 3D objects for which we generated *.dat* files. This enabled us (a) to export and use HP-GL output files, and (b) to display not only solid objects but also individual lines, such as the

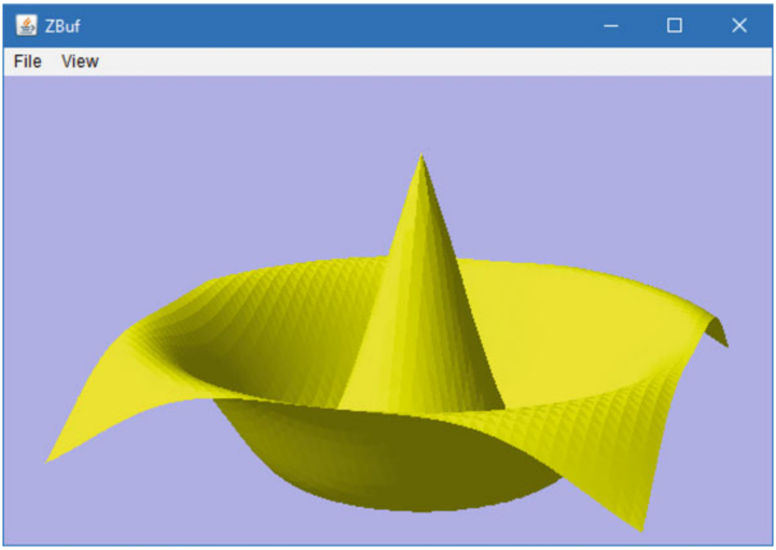


Fig. D.15 The object of Fig. D.12, showing faces instead of lines

coordinate axes in Fig. D.12. Recall that we can also use the generated data files as input for our programs *ZBuf.java* or *Painter.java*, to display the faces of these objects. Individual line segments are then simply omitted. Rather than doing this for all 3D objects in this appendix, we do this only for our last example, as Fig. D.15 shows.

Appendix E: Hints and Solutions to Exercises

1.1 Numbers of pixels:

```
g.drawLine(10, 20, 100, 50); // 100 - 10 + 1 = 91 pixels
g.drawRect(10, 10, 8, 5); // 2 * 8 + 2 * 5 = 26 pixels
g.fillRect(10, 10, 8, 5); // 8 * 5 = 40 pixels
```

1.2 Program to draw many squares:

```
//ManySq.java: This program draws n x n sets, each
//consisting of k squares, arranged as on a chessboard.
//Each edge is divided into two parts with ratio
//(1 - q) : q. The values of n, k and q are program arguments.
import java.awt.*;
import java.awt.event.*;

public class ManySq extends Frame {
   public static void main(String[] args) {
      if (args.length != 3) {
         System.out.println("Supply n, k and q as arguments");
         System.exit(1);
      }
      int n = Integer.valueOf(args[0]).intValue(),
         k = Integer.valueOf(args[1]).intValue();
      float q = Float.valueOf(args[2]).floatValue();
      new ManySq(n, k, q);
   }
```


L. Ammeraal, K. Zhang, *Computer Graphics for Java Programmers*,
DOI 10.1007/978-3-319-63357-2

```
   ManySq(int n, int k, float q) {
      super("ManySq: Many squares");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e)
            {System.exit(0);}
      });
      add("Center", new CvManySq(n, k, q));
      setSize(600, 400);
      setVisible(true);
   }
}

class CvManySq extends Canvas {
   int centerX, centerY, n, k;
   float p0, q0;

   CvManySq(int nn, int kk, float qq) {
      n = nn; k = kk; q0 = qq; p0 = 1 - q0;
   }

   int iX(float x) {return Math.round(centerX + x);}
   int iY(float y) {return Math.round(centerY - y);}

   public void paint(Graphics g) {
      Dimension d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1,
          minMaxXY = Math.min(maxX, maxY);
      centerX = maxX / 2; centerY = maxY / 2;

      float r = 0.45F * minMaxXY / n;
      for (int x = 0; x < n; x++) {
         for (int y = 0; y < n; y++) {
            float xCnew = (2 * x - n + 1) * r,
                  yCnew = (2 * y - n + 1) * r,
                  xA, yA, xB, yB, xC, yC, xD, yD,
                  xA1, yA1, xB1, yB1, xC1, yC1, xD1, yD1,
                  p = p0, q = q0;
            if (x % 2 + y % 2 == 1) {p = q0; q = p0;}
            xA = xD = xCnew - r; xB = xC = xCnew + r;
            yA = yB = yCnew - r; yC = yD = yCnew + r;
            for (int i = 0; i < k; i++) {
               g.drawLine(iX(xA), iY(yA), iX(xB), iY(yB));
               g.drawLine(iX(xB), iY(yB), iX(xC), iY(yC));
               g.drawLine(iX(xC), iY(yC), iX(xD), iY(yD));
               g.drawLine(iX(xD), iY(yD), iX(xA), iY(yA));
               xA1 = p * xA + q * xB; yA1 = p * yA + q * yB;
               xB1 = p * xB + q * xC; yB1 = p * yB + q * yC;
```

```
                xC1 = p * xC + q * xD; yC1 = p * yC + q * yD;
                xD1 = p * xD + q * xA; yD1 = p * yD + q * yA;
                xA = xA1; xB = xB1; xC = xC1; xD = xD1;
                yA = yA1; yB = yB1; yC = yC1; yD = yD1;
             }
          }
       }
    }
 }
```

1.3 To draw all edges as exactly straight lines and to make the vertices of inner squares lie exactly on the edges of their surrounding squares, use device coordinates, starting with a pair of very small squares (◊ and O), and making the squares of each next pair exactly twice as large as those of the preceding pair.

1.4 The radius r of the circumscribed circles for the hexagons is supplied by the user. Based on this radius r, the following fragment (in which the variable names are self-explanatory) may be helpful:

```
int iX(float x){return Math.round(centerX + x/pixelSize);}
int iY(float y){return Math.round(centerY - y/pixelSize);}

void drawLine(Graphics g, float xA, float yA, float xB, float yB) {
   g.drawLine(iX(xA), iY(yA), iX(xB), iY(yB));
}
...

float halfr = r/2, horpitch = 1.5F * r,
  w = r * (float)Math.sqrt(3), h = w/2, marginleft, marginbottom;
int nhor = (int)Math.floor((rWidth - 2 * r) / horpitch) + 1,
    nvert = (int)Math.floor(rHeight/w);
marginleft = -rWidth/2 + 0.5F * (rWidth - halfr - nhor * horpitch);
marginbottom = -rHeight/2 + 0.5F * (rHeight - nvert * w);
for (int i=0; i<nhor; i++) {
   float x = marginleft + r + i * horpitch,
    y0 = marginbottom + (1 + i % 2) * h; // center of lowest hexagon
   int m = nvert - i % 2;
// There will be nvert hexagons in each column for i = 0, 2, 4, ...
// while there will be nvert - 1 in each column for i = 1, 3, 5, ...
// Special case: if nvert = 1 and nhor > 1, then x is increased by
// horpitch/2 because otherwise there will be an empty column on the
// right.
   if (nvert == 1 && nhor > 1)
      x += horpitch/2;
```

```
for (int j=0; j<m; j++) {
   float y = y0 + j * w;
   drawLine(g, x + halfr, y + h, x - halfr, y + h);
   drawLine(g, x - halfr, y + h, x - r, y);
   drawLine(g, x - r, y, x - halfr, y - h);
   ...
```

1.5 We begin by computing the length

$$L = \sqrt{{u_1}^2 + {u_2}^2}$$

where

$$u_1 = x_B - x_A$$
$$u_2 = y_B - y_A$$

Since there should be a dash, not a gap, at each endpoint, and we use gap widths that are about equal to *dashLength*, we use the equality

$$L = (2n - 1) \times dashLength$$

to compute n, the number of dashes. Writing $h_1 = u_1/(2n - 1)$ and $h_2 = u_2/(2n - 1)$, and denoting the dashes by $i = 0, 1, \ldots, n - 1$, we draw dash i as a straight line with endpoints $(x_A + 2ih_1, y_A + 2ih_2)$ and $((x_A + (2i + 1)h_1, y_A + (2i + 1)h_2)$.

1.6 Design the data structure in such a way that it is possible to be extended to 3-dimensional for Exercise 6.10. Draw the interface of the Tetris game and every component, using the Java *drawLine*, *drawString* and *fillShape* methods. Divide the canvas into 10 × 20 grids, each representing one square. Moving a shape on the canvas is then equivalent to moving the colored squares representing the shape by one grid downward at a time.

2.1 After rotating the vector $\mathbf{v} = (v_1, v_2)$ through 90° counter-clockwise, we obtain the vector $(-v_2, v_1)$. Setting $\mathbf{v} = (v_1, v_2) = (x_B - x_A, y_B - y_A)$, we can therefore find the points D and C by adding $(-v_2, v_1)$ to the coordinates of A and B, respectively.

2.2 To determine the position of P relative to the triangle ABC, we first test whether P lies on one of the three sides of the triangle, using the method *onSegment*, discussed in Sect. 2.7. If this is not the case, we test whether P lies inside ABC, using the method *insideTriangle* of Sect. 2.5. To do this properly, we need to know the orientation of A, B and C, for which we can use the method *ccw* of Sect. 2.4. If this orientation is clockwise, we use C, B, and A, in that order, as the first three arguments of *insideTriangle*, so that the orientation of these arguments is counter-clockwise, as required. If P lies neither on a triangle side nor inside the triangle, it lies outside it.

2.3 Section 2.9 shows how to compute the distance between a point and a line. We perform this computation three times to determine which of the three triangle sides AB, BC and CA (or rather the infinite lines through these sides) lies closest to point P. We then use the method *projection* of Sect. 2.8 to compute the projection P′ of P on the triangle side in question (or on an extension of it). We draw both the triangle and the line segment PP′. If the projection point P′ lies on an extension of a side, we also connect this point to the side (AB, BC or CA), to indicate clearly which of the three lines has been used. For example, if P′ lies on an extension of BC (not between B and C), we can draw P′B.

2.4 Using the vector $\mathbf{AB} = \mathbf{u} = (u_1, u_2) = (x_B - x_A, y_B - y_A)$ and the parameter λ, we can represent the line through A and B by the following vector form:

$$\mathbf{A} + \lambda \mathbf{u}$$

Similarly, with $\mathbf{CD} = \mathbf{v} = (v_1, v_2) = (x_D - x_C, y_D - y_C)$ and parameter μ, the line through C and D is represented by

$$\mathbf{C} + \lambda \mathbf{v}$$

We find the intersection point S by solving

$$\mathbf{A} + \lambda \mathbf{u} = \mathbf{C} + \lambda \mathbf{v}$$

for λ (rewriting this vector equation as a system of two linear equations, using the x- and y-coordinates of A and C as well as u_1, u_2, v_1 and v_2). We then use the value of λ computed in this way to find

$$\mathbf{S} = \mathbf{A} + \lambda \mathbf{u}$$

When deriving the desired value of λ, we will have to perform a division by the expression $u_2 v_1 - u_1 v_2$ (which is a determinant). If this determinant is zero, the lines AB and CD do not have a unique intersection point because these lines are parallel or coinciding. Since the points A, B, C and D are obtained by clicking and there are (very small) rounding-off errors, we had better replace the condition

$$determinant = 0$$

with this one:

$$| \ determinant \ | \leq epsilon$$

where *epsilon* is some very small positive value. To make independent of the units of length that are used and in view of the way the determinant is computed, a reasonable value is

$$epsilon = 10^{-3}\left({u_1}^2 + {u_2}^2 + {v_1}^2 + {v_2}^2\right)$$

2.5 To construct the bisector of angle B, we can compute the two vectors $\mathbf{u} = \mathbf{BA}/|\mathbf{BA}|$ and $\mathbf{v} = \mathbf{BC}/|\mathbf{BC}|$. We can view these vectors as arrows starting at B and pointing to A and C, respectively. Since both $\mathbf{u}$ and $\mathbf{v}$ have length 1, the sum vector

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

can then be regarded as another arrow starting at B but lying on the desired bisector, so that, with parameter α, the vector form

$$\mathbf{B} + \alpha\mathbf{w}$$

denotes the bisector of angle B. The intersection point D of this bisector and triangle side AC can then be found in the same way as in Exercise 2.4.

2.6 Using $\mathbf{AB} = \mathbf{u} = (u_1, u_2) = (x_B - x_A, y_B - y_A)$ and $\mathbf{v} = (v_1, v_2) = (-u_2, u_1)$, we can write the following vector form for the perpendicular bisector of AB:

$$\mathbf{A} + 0.5\mathbf{u} + \lambda\mathbf{v} \tag{F.1}$$

To find the circumcenter D of triangle ABC, write a similar vector form, say,

$$\mathbf{B} + 0.5\mathbf{w} + \mu\mathbf{t} \tag{F.2}$$

for the perpendicular bisector of BC. You can then find the intersection of these two lines by solving the vector equation

$$\mathbf{A} + 0.5\mathbf{u} + \lambda\mathbf{v} = \mathbf{B} + 0.5\mathbf{w} + \mu\mathbf{t}$$

for λ (or μ). Then the circumcenter D is found as the point of intersection by using this value λ in Eq. (F.1). After computing the radius $r = |\mathbf{AD}|$, and using the methods *iX* and *iY* to convert real logical coordinates into integer device coordinates, you can draw the circle through A, B and C by writing

```
int xLeft = iX(xD - r), xRight = iX(xD + r),
    yTop = iY(yD + r), yBottom = iY(yD - r);
g.drawOval(xLeft, yTop, xRight - xLeft, yBottom - yTop);
```

2.7 Compute the center C and the radius r of the circle through P, Q and R (see Exercise 2.1). Although there is a method *drawArc* in Java, this is not suitable for our present purpose because it requires angles to be specified (in degrees) as integers; especially if r is large, this may cause too large rounding-off errors with regard to both endpoints of the arc. We therefore simply use a great many straight line segments. We can do this by using *Point2D* objects for C, P, Q and R (see Sect. 1.4). Taking the orientation of P, Q and R into account by means of the method *area*2 of class *Tools*2*D* (see Sect. 2.3), we can write:

```
double alpha = Math.atan2(P.y - C.y, P.x - C.x),
       beta = Math.atan2(R.y - C.y, R.x - C.x);
if (Tools2D.area2(P, Q, R) > 0)
    arcCcw(g, C, r, alpha, beta);
else
    arcCcw(g, C, r, beta, alpha);
```

The method *arcCcw*, used here, is listed below. Working counter-clockwise, it draws the arc with start and end angles *alpha* and *beta* and belonging to the circle with center C and radius *r*:

```
void arcCcw(Graphics g, Point2D C, double r,
   double alpha, double beta) {
   double pi2 = 2 * Math.PI, delta = beta - alpha;
   // Reduce delta to the interval [0, 2pi):
   delta = (delta + pi2) % pi2;
   int X0=0, Y0=0,                // Arc length = r * delta
       n = (int)Math.ceil(r * delta / 0.02); // 0.02 = rWidth/500
   double theta = delta / n;
   for (int i=0; i<=n; i++) {
      double phi = alpha + i * theta,
        x = C.x + r * Math.cos(phi),
        y = C.y + r * Math.sin(phi);
      int X = iX((float)x), Y = iY((float)y);
      if (i > 0) g.drawLine(X0, Y0, X, Y);
      X0 = X; Y0 = Y;
   }
}
```

2.8 We can use the first of the two methods *projection* of Sect. 2.8 to find the projection D′ of D on AB. Since the center M of the circular arc lies on the bisector of the angle ABC, we compute the unit vectors $\mathbf{u} = \mathbf{BA}/|\mathbf{BA}|$ and $\mathbf{v} = \mathbf{BC}/|\mathbf{BC}|$ and $\mathbf{w} = (\mathbf{u} + \mathbf{v})/|\mathbf{u} + \mathbf{v}|$, which start at B and point to A, C and M, respectively. We now have to find a scale factor λ, so that $\mathbf{BM} = \lambda$w. Since the cosine of the angle D′BM is equal to $\mathbf{w} \cdot \mathbf{v}$ and using $\mu = |\mathbf{BD}'|$, we can compute $\lambda = \mu / (\mathbf{w} \cdot \mathbf{v})$. We then find the endpoint E of the arc on BC and the center M as follows: $\mathbf{E} = \mathbf{B} + \mu\mathbf{v}$, $\mathbf{M} = \mathbf{B} + \lambda\mathbf{w}$. Obviously, the radius of the arc is $r = |\mathbf{MD}'|$. We can now compute the start and end angles α and β and draw the arc by choosing between two calls to the method *arcCcw*, depending on the orientation of the points A, B and C (see Exercise 2.2).

2.9 Refer to Exercise 2.3 for bisectors of angles and to Exercise 2.1 for the intersection of two lines. This will provide you with the centers of the four circles. You can use the radius of each circle as the distance of its center to one of its tangents. Recall that we have discussed the distance of points to lines in Sect. 2.9.

2.10 Use vector $\mathbf{AB} = (u_1, u_2)$ to find the points $\mathrm{D} = \mathrm{A} + (-u_2, u_1)$, $\mathrm{C} = \mathrm{D} + (u_1, u_2)$ and $\mathrm{E} = \mathrm{D} + 0.5\{(u_1, u_2) + (-u_2, u_1)\}$.

3.1 $M = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ x_\mathrm{C}(1 - s_x) & y_\mathrm{C}(1 - s_y) & 1 \end{bmatrix}$

3.2 Similar to Exercise 3.1.

3.3 For shearing a set of points with reference to point C, we replace the shearing equations at the end of Sect. 3.2 with the similar ones

$$\begin{aligned} x' - x_\mathrm{C} &= (x - x_\mathrm{C}) + a(y - y_\mathrm{C}) \\ y' - y_\mathrm{C} &= (y - y_\mathrm{C}) \end{aligned}$$

which reduces to $x' = x + a(y - y_\mathrm{C})$, $y' = y$.

The shearing operation will transform the circle into an ellipse with a non-horizontal axis, so that we cannot use the Java method *drawOval*. Therefore, for some large value of n, we approximate a circle with center $\mathrm{C}(x_\mathrm{C}, y_\mathrm{C})$ and radius r by computing the following n points (x_i, y_i) of this circle:

$$x_i = x_\mathrm{C} + r \cos i\theta$$

$$y_i = y_\mathrm{C} + r \sin i\theta$$

where $\theta = 2\pi/n$ and $i = 0, 1, ..., n - 1$. Instead of immediately connecting these points by straight lines, which would produce the circle, we first subject each x_i to the above shearing formula.

3.4 Just compute the product AA^{-1} to obtain the identity matrix I. For example, the upper-left element of this product is equal to the inner product of the first row of A and the first column of A^{-1}, which is $a_{11}(a_{22}/D) + a_{12}(-a_{21}/D) = D/D = 1$.

3.5 If there are many points for which we have to check whether they lie within a single triangle (or on an edge of it), the method *insideTriangle* of the following class is more efficient than the one discussed in Sect. 2.5, since most of the work is done here by the constructor, which need to be called only once for that triangle:

```
class TriaTest {
   private Point2D C;
   private double a1, a2, b1, b2, c1, c2, d1, d2, det;

   TriaTest(Point2D A, Point2D B, Point2D C) {
      this.C = C;
```

```
      a1 = A.x - C.x; a2 = A.y - C.y;
      b1 = B.x - C.x; b2 = B.y - C.y;
      det = a1 * b2 - b1 * a2;
      if (det != 0) {
          c1 = b2/det; c2 = -a2/det;
          d1 = -b1/det; d2 = a1/det;
      }
   }

   double area2(){return det;}

   boolean insideTriangle(Point2D P) {
      double p1 = P.x - C.x, p2 = P.y - C.y,
         lambda, mu;
      return (lambda = p1 * c1 + p2 * d1) >= 0 &&
          (mu = p1 * c2 + p2 * d2) >= 0 &&
          lambda + mu <= 1;
   }
}
```

4.1 Adapt the Java program for Bresenham's algorithm by drawing pixels from both of the endpoints towards the middle of the line. Either calculate where the middle point is beforehand or check on-the-fly (that is, within the loop) when the two pixels merge in the middle. There may be one or two middle points depending on whether the line consists of odd or even number of pixels. If there is only one, it is a good idea to draw this pixel after exiting the loop. Check if your solution also works correctly for very short lines, consisting of one or two pixels. You solution should be very general in that it works for any two endpoints P and Q.

4.2 You should add a second for-loop in which the roles of x and y are interchanged. For example, the calls to *putPixel* should have ++y as their third argument instead of ++x as their second. If $|y_Q - y_P| \leq |x_Q - x_P|$, the first loop should be executed; otherwise the second.

4.4 The following program produces only Fig. 4.19. You should extend it, enabling the user to specify the two endpoints of a line segment and both the center and the radius of a circle.

```
// Bresenham.java: Bresenham algorithms for lines and circles
//                 demonstrated by using superpixels.
import java.awt.*;
import java.awt.event.*;

public class Bresenham extends Frame {
   public static void main(String[] args) {new Bresenham();}
```

```
   Bresenham() {
      super("Bresenham");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e)
         {System.exit(0);}
      });
      setSize(340, 230);
      add("Center", new CvBresenham());
      setVisible(true);
   }
}

class CvBresenham extends Canvas {
   float rWidth = 10.0F, rHeight = 7.5F, pixelSize;
   int centerX, centerY, dGrid = 10, maxX, maxY;

   void initgr() {
      Dimension d = getSize();
      maxX = d.width - 1; maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
   }

   int iX(float x) {return Math.round(centerX + x / pixelSize);}
   int iY(float y) {return Math.round(centerY - y / pixelSize);}

   void putPixel(Graphics g, int x, int y) {
      int x1 = x * dGrid, y1 = y * dGrid, h = dGrid / 2;
      g.drawOval(x1 - h, y1 - h, dGrid, dGrid);
   }

   void drawLine(Graphics g, int xP, int yP, int xQ, int yQ) {
      int x = xP, y = yP, D = 0, HX = xQ - xP, HY = yQ - yP,
          c, M, xInc = 1, yInc = 1;
      if (HX < 0) {xInc = -1; HX = -HX;}
      if (HY < 0) {yInc = -1; HY = -HY;}
      if (HY <= HX) {
         c = 2 * HX; M = 2 * HY;
         for (;;) {
            putPixel(g, x, y);
            if (x == xQ) break;
            x += xInc;
            D += M;
            if (D > HX) {y += yInc; D -= c;}
         }
```

```
      } else {
         c = 2 * HY; M = 2 * HX;
         for (;;) {
            putPixel(g, x, y);
            if (y == yQ) break;
            y += yInc; D += M;
            if (D > HY) {x += xInc; D -= c;}
         }
      }
   }

   void drawCircle(Graphics g, int xC, int yC, int r) {
      int x = 0, y = r, u = 1, v = 2 * r - 1, E = 0;
      while (x < y) {
         putPixel(g, xC + x, yC + y); // NNE
         putPixel(g, xC + y, yC - x); // ESE
         putPixel(g, xC - x, yC - y); // SSW
         putPixel(g, xC - y, yC + x); // WNW
         x++; E += u; u += 2;
         if (v < 2 * E) {y--; E -= v; v -= 2;}
         if (x > y) break;
         putPixel(g, xC + y, yC + x); // ENE
         putPixel(g, xC + x, yC - y); // SSE
         putPixel(g, xC - y, yC - x); // WSW
         putPixel(g, xC - x, yC + y); // NNW
      }
   }

   void showGrid(Graphics g) {
      for (int x = dGrid; x <= maxX; x += dGrid)
         for (int y = dGrid; y <= maxY; y += dGrid)
            g.drawLine(x, y, x, y);
   }

   public void paint(Graphics g) {
      initgr();
      showGrid(g);
      drawLine(g, 1, 1, 12, 5);
      drawCircle(g, 23, 10, 8);
   }
}
```

4.5 Since an unknown number of curve segments are to be dealt with, we can use the Java concept of *Vector*, as we have also done in Sect. 1.4 and elsewhere to store *Point2D* objects representing the vertices of a polygon. In this case it makes sense to define a class *CurveSegment* and to use a *Vector* of *CurveSegment* objects, as this fragment shows:

```
class CurveSegment {
   Point2D[] P;
   CurveSegment(Point2D[] P){this.P = P;}
   ...
}
```

Writing

```
Vector curves = new Vector();
```

and using the array *P*, declared as

```
Point2D[] P = new Point2D[4];
```

containing the most recent four points, as we did in program *Bezier.java* in Sect. 4.6, we can add a new curve segment to *curves* by writing

```
curves.addElement(new CurveSegment(P));
```

The object *curves* can store several curves, each consisting of some consecutive elements.

4.6 In program *Bspline.java* of Sect. 4.7, pressing a key is interpreted as a signal to terminate the process of extending the curve. Insert the line

```
char ch = evt.getKeyChar();
```

in the method *keyTyped* so that you can use the character *ch* to differentiate between different characters entered by the user and to use them as commands.

Use a *Vector* element for each array representing a curve. Recall that we have used the statement

```
V.copyInto(P);
```

in the *paint* method of program *Bspline.java*, to copy the *Vector* object *V* into the array *P*. Using a different Vector *object*, say, *curves*, we can now add the array *P* to *curves*. The deletion of the last curve, as required by the *d* command, is then implemented as

```
curves.setSize(curves.size()-1);
```

4.7 Represent the grid on the screen by drawing ten equidistant horizontal lines that intersect ten equidistant vertical lines. If the user clicks on (or near) a point of intersection of these lines, transform the device coordinates to gridpoint coordinates, ranging from 0 through 9, and use these gridpoint coordinates as to select P and Q. Draw the line PQ after Q has been defined. On the right of all these horizontal and vertical lines, display the strings

algorithm[0], *algorithm*[1], ..., *algorithm*[7]

below each other, where the array *algorithm* is defined and initialized as follows:

```
String[] algorithm = {
  "int x=xP,y=yP,d=0,dx=xQ-xP,c=2*dx,", // 0
  "    m=2*(yQ-yP);",                    // 1
  "for (;;) {",                          // 2
  "  putPixel(g, x, y);",                // 3
  "  if (x == xQ) break;",               // 4
  "  x++; d += m;",                      // 5
  "  if (d > dx){y++; d -= c;}",         // 6
  "}"};                                  // 7
```

As soon as the user has defined point Q, the line stored as *algorithm*[3] should be highlighted, indicating that the call to *putPixel* is about to be executed. You can realize this by using a variable, say *i*, indicating which of the above eight program lines (if any) should be displayed in red (or equal to, say, -1 if all program lines are to appear in black). All lines *algorithm*[*j*] with $j \neq i$ are displayed in black. You can use a switch statement to test the value of *i* in a method *stepPressed*. For example, you can write a fragment of the following form in the constructor of your canvas class:

```
addMouseListener(new MouseAdapter() {
   public void mousePressed(MouseEvent evt) {
      // When the mouse is clicked, determine where
      // it is on the screen and do the appropriate
      // action, if any.
      int xClick = 0, yClick = 0;
      // Get the coordinates
      xClick = evt.getX();
      yClick = evt.getY();
      // Check to see if STEP button was pressed
      if (point (xClick, yClick) lies within the
         rectangle representing the Step button)
         stepPressed();
      else
      if (point (xClick, yClick) lies within grid area {
          ...
      }
       repaint();
   }
});
```

In the switch statement just mentioned, you should execute actions defined in the program line (stored in the *algorithm* array) that was previously highlighted and update the variable *i* mentioned above. Your method *paint* will use this variable *i* to display the correct program line in red and the others in black.

5.1 As Fig. 5.11 shows, nine cube edges are visible and three are invisible. In the *paint* method of program *CubePers.java*, replace the calls to the method *line* with this fragment:

```
// Visible edges:
line(g, 0, 1); line(g, 1, 5); line(g, 5, 4); line(g, 4, 0);
line(g, 1, 2); line(g, 2, 6); line(g, 6, 7); line(g, 7, 4);
line(g, 5, 6);

// Invisible edges:
g.setColor(Color.blue);
line(g, 0, 3); line(g, 3, 2); line(g, 3, 7);
```

If you did Exercise 1.5 and have a class *Lines*, containing the method *dashedLine*, available, you may be able to replace the last two of the above lines with

```
dash(g, 0, 3); dash(g, 3, 2); dash(g, 3, 7);
```

while adding the following method to the class *CvCubePers*:

```
void dash(Graphics g, int i, int j) {
   Point2D P = obj.vScr[i], Q = obj.vScr[j];
  Lines.dashedLine(g, iX(P.x), iY(P.y), iX(Q.x), iY(Q.y), 8);
}
```

Figure E.1 shows the result of this solution with dashed lines.

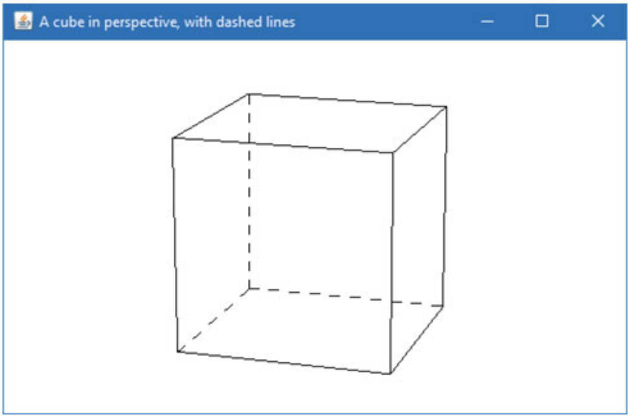


Fig. E.1 *Dashed lines* representing invisible edges

5.2 There are two *fillPolygon* methods: one taking a *Polygon* object as an argument and the other taking two arrays *x* and *y* instead. In either case, do not forget to convert logical to device coordinates, using the methods *iX* and *iY*. Use *setColor*, with different colors before each of the three calls to *fillPolygon*.

5.3 To replace Fig. 5.11, begin by sketching two cubes, say, one on either side of the *x*O*z*-plane, and by assigning the numbers 0–7 to the vertices of the first and 8–15 to those of second cube. With this sketch, and using arrays *w* and *vScr* with length 16 instead of 8, you can easily update the program as requested. Remember to increase the value of *objSize*, which is used to compute both the object distance ρ and the screen distance *d*.

5.4 The following program demonstrates the principle of animation (with double buffering) for a simple case: a line segment is rotated about one of its endpoints, which is the center of the canvas. Every 20 ms, the angle α is increased by 0.01 radians and the line from the origin O (in the center of the canvas) to point ($r \cos \alpha$, $r \sin \alpha$) is drawn. The effect is that of a running clock with only one hand:

```
// Anim.java: Animation with double buffering.
import java.awt.*;
import java.awt.event.*;

public class Anim extends Frame {
   public static void main(String[] args) {new Anim();}

   Anim() {
      super("Animation (double buffering)");
      addWindowListener(new WindowAdapter() {
       public void windowClosing(WindowEvent e) {System.exit(0);}
      });
      add("Center", new CvAnim());
      Dimension dim = getToolkit().getScreenSize();
      setSize(dim.width / 2, dim.height / 2);
      setLocation(dim.width / 4, dim.height / 4);
      setVisible(true);
   }
}

class CvAnim extends Canvas implements Runnable {
  float rWidth = 10.0F, rHeight = 10.0F, xC, yC, pixelSize;
  int centerX, centerY, w, h;
  Dimension d;
  Image image;
  Graphics gImage;

  float alpha = 0;
  Thread thr = new Thread(this);
```

```
   public void run() {
      try {
         for (;;) {
             alpha += 0.01;
             repaint();
             Thread.sleep(20);
         }
      } catch (InterruptedException e) {
      }
   }

   CvAnim() {
      thr.start();
   }

   void initgr() {
      d = getSize();
      int maxX = d.width - 1, maxY = d.height - 1;
      pixelSize = Math.max(rWidth / maxX, rHeight / maxY);
      centerX = maxX / 2; centerY = maxY / 2;
      xC = rWidth / 2; yC = rHeight / 2;
   }

   int iX(float x) {return Math.round(centerX + x / pixelSize);}
   int iY(float y) {return Math.round(centerY - y / pixelSize);}
   public void update(Graphics g) {paint(g);}

   public void paint(Graphics g) {
      initgr();
      if (w != d.width || h != d.height) {
          w = d.width; h = d.height;
          image = createImage(w, h);
          gImage = image.getGraphics();
      }
      float r = 0.8F * Math.min(xC, yC),
           x = r * (float) Math.cos(alpha),
           y = r * (float) Math.sin(alpha);
      gImage.clearRect(0, 0, w, h);
      // Every 20 ms, the following line is drawn.
      // Each time, its endpoint (x, y) is a
      // different point on a circle:
      gImage.drawLine(iX(0), iY(0), iX(x), iY(y));
      g.drawImage(image, 0, 0, null);
   }
}
```

5.5 The following program produces two rotating cubes, illustrated by 5.15. Remember, this program works only if the class file *Rota3D.class* (see Sect. 3.9) is available in the current directory:

```
// CubRot2.java: Two rotating cubes with double buffering.
//   Uses: Point2D (Section 1.4),
//         Point3D, Rota3D (Section 3.9)
import java.awt.*;
import java.awt.event.*;

public class CubRot2 extends Frame {
   public static void main(String[] args) {new CubRot2();}

   CubRot2() {
      super("Rotating cubes (double buffering)");
      addWindowListener(new WindowAdapter() {
         public void windowClosing(WindowEvent e) {
            System.exit(0);
         }
      });
      add("Center", new CvCubRot2());
      Dimension dim = getToolkit().getScreenSize();
      setSize(3 * dim.width / 4, dim.height / 2);
      setLocation(dim.width / 8, dim.height / 4);
      setVisible(true);
   }
}

class CvCubRot2 extends Canvas implements Runnable {
   int centerX, centerY, w, h;
   Obj2 obj = new Obj2();
   Image image;
   Graphics gImage;
   double alpha = 0;
   Thread thr = new Thread(this);

   public void run() {
      try {
         for (;;) {
            alpha += 0.01;
            repaint();
            Thread.sleep(20);
         }
      } catch (InterruptedException e) {
      }
   }
```

```
   CvCubRot2() {thr.start();}
   public void update(Graphics g) {paint(g);}

   int iX(float x) {return Math.round(centerX + x);}

   int iY(float y) {return Math.round(centerY - y);}

   void line(int i, int j) {
      Point2D P = obj.vScr[i], Q = obj.vScr[j];
      gImage.drawLine(iX(P.x), iY(P.y), iX(Q.x), iY(Q.y));
   }

   public void paint(Graphics g) {
      Dimension dim = getSize();
      int maxX = dim.width - 1, maxY = dim.height - 1;
      centerX = maxX / 2; centerY = maxY / 2;
      int minMaxXY = Math.min(maxX, maxY);
      obj.d = obj.rho * minMaxXY / obj.objSize;
      obj.rotateCube(alpha);
      obj.eyeAndScreen();
      if (w != dim.width || h != dim.height) {
         w = dim.width; h = dim.height;
         image = createImage(w, h);
         gImage = image.getGraphics();
      }
      gImage.clearRect(0, 0, w, h);
      // Horizontal edges at the bottom:
      line(0, 1); line(1, 2); line(2, 3); line(3, 0);
      // Horizontal edges at the top:
      line(4, 5); line(5, 6); line(6, 7); line(7, 4);
      // Vertical edges:
      line(0, 4); line(1, 5); line(2, 6); line(3, 7);
      // Same for second cube:
      line(8, 9); line(9, 10); line(10, 11); line(11, 8);
      // Horizontal edges at the top:
      line(12, 13); line(13, 14); line(14, 15);
      line(15, 12);
      // Vertical edges:
      line(8, 12); line(9, 13); line(10, 14); line(11, 15);
      g.drawImage(image, 0, 0, null);
   }
}
```

```
class Obj2 { // Contains 3D object data for two cubes
   float rho, theta = 0F, phi = 1.3F, d;
   Point3D[] s, w; // World coordinates
   Point2D[] vScr; // Screen coordinates
   float v11, v12, v13, v21, v22, v23,
         v32, v33, v43, // Elements of viewing matrix V.
         xe, ye, ze, objSize = 8;

   Obj2() {
      s = new Point3D[16]; // Start situation
      w = new Point3D[16]; // After rotation
      vScr = new Point2D[16];
      // Bottom surface:
      s[0] = new Point3D(1, -3, -1);
      s[1] = new Point3D(1, -1, -1);
      s[2] = new Point3D(-1, -1, -1);
      s[3] = new Point3D(-1, -3, -1);
      // Top surface:
      s[4] = new Point3D(1, -3, 1);
      s[5] = new Point3D(1, -1, 1);
      s[6] = new Point3D(-1, -1, 1);
      s[7] = new Point3D(-1, -3, 1);
      // Bottom surface:
      s[8] = new Point3D(1, 1, -1);
      s[9] = new Point3D(1, 3, -1);
      s[10] = new Point3D(-1, 3, -1);
      s[11] = new Point3D(-1, 1, -1);
      // Top surface:
      s[12] = new Point3D(1, 1, 1);
      s[13] = new Point3D(1, 3, 1);
      s[14] = new Point3D(-1, 3, 1);
      s[15] = new Point3D(-1, 1, 1);
      rho = 15; // For reasonable perspective effect
   }

   void rotateCube(double alpha) {
      Rota3D.initRotate(s[0], s[4], alpha);
      for (int i = 0; i < 8; i++)
         w[i] = Rota3D.rotate(s[i]);
      Rota3D.initRotate(s[13], s[9], 2 * alpha);
      for (int i = 8; i < 16; i++)
         w[i] = Rota3D.rotate(s[i]);
   }
```

```
    void initPersp() {
       float costh = (float) Math.cos(theta),
             sinth = (float) Math.sin(theta),
             cosph = (float) Math.cos(phi),
             sinph = (float) Math.sin(phi);
      v11 = -sinth; v12 = -cosph * costh; v13 = sinph * costh;
      v21 = costh;  v22 = -cosph * sinth; v23 = sinph * sinth;
                    v32 = sinph;          v33 = cosph;
                                          v43 = -rho;
    }

    void eyeAndScreen() {
       initPersp();
       for (int i = 0; i < 16; i++) {
          Point3D P = w[i];
          float x = v11 * P.x + v21 * P.y;
          float y = v12 * P.x + v22 * P.y + v32 * P.z;
          float z = v13 * P.x + v23 * P.y + v33 * P.z + v43;
          Point3D Pe = new Point3D(x, y, z);
          vScr[i] =
             new Point2D(-d * Pe.x / Pe.z,
                         -d * Pe.y / Pe.z);
       }
    }
}
```

5.6 A program to generate an open book is shown below. It was executed twice (with $n = 4$ and $n = 150$) to produce the two open books of Fig. 5.26. Refer to the analysis of Exercise 5.7 below for the way we design this type of programs.

```
// BookView.java: Generating a data file for an open book.
import java.io.*;

public class BookView
{ public static void main(String[] args)
     throws IOException {
     if (args.length != 4) {
        System.out.println(
        "Supply nr of sheets, width, height and file name\n"+
        "as program arguments.");
        System.exit(1);
     }
     int n;
     float w, h;
```

```
      FileWriter fw;
      n = Integer.valueOf(args[0]).intValue();
      w = Float.valueOf(args[1]).floatValue();
      h = Float.valueOf(args[2]).floatValue();
      fw = new FileWriter(args[3]);
      int spineTop = 1, spineBottom = 2, outerTop, outerBottom;
      float theta = (float)Math.PI/(n - 1);
      float xTop = 0, xBottom = h;
      fw.write(spineTop + " " + xTop + " 0 0\r\n");
      fw.write(spineBottom + " " + xBottom + " 0 0\r\n");
      for (int i=0; i<n; i++) {
         float phi = i * theta,
            y = w * (float)Math.cos(phi),
            z = w * (float)Math.sin(phi);
         outerTop = 2 * i + 3; outerBottom = outerTop + 1;
         fw.write(outerTop + " " + xTop + " " +
            y + " " + z + "\r\n");
         fw.write(outerBottom + " " + xBottom + " " +
            y + " " + z + "\r\n");
      }
      fw.write("Faces:\r\n");
      for (int i=0; i<n; i++) {
         outerTop = 2 * i + 3; outerBottom = outerTop + 1;
         fw.write(spineTop + " " + spineBottom + " "
            + outerBottom + " " + outerTop + ".\r\n");
         fw.write(spineTop + " " + outerTop + " "
            + outerBottom + " " + spineBottom + ".\r\n");
      }
      fw.close();
   }
}
```

5.7 Before writing the program code we have to assign numbers to vertices and find mathematical expression for the x-, y- and z-coordinates of these vertices. We will now discuss how this can be done for a sphere, but the same approach applies to any program that generates 3D data files.

The model of a sphere in question has two poles; let us assign vertex number 1 to the north pole. Since it is given that there are n horizontal slices, there will be $n - 1$ horizontal planes between them, each corresponding with a horizontal circle, or line of latitude, on the sphere. There will also be $2n$ vertical circles, or lines of longitude. Every vertex (other than the two poles) of our sphere model is a point of intersection of such a horizontal and a vertical circle. As for the faces, $2 \times 2n$ of them are triangles at the two poles. There are $n - 2$ remaining horizontal slices, so that the number of remaining faces is equal to $(n - 2) \cdot (2n) = 2n(n - 2)$. Each of these is a parallelogram with two horizontal edges. Altogether, there are $4n + 2n(n - 2) = 2n^2$ faces, and, as we will see below, $2(n^2 - n + 1)$ vertices.

We will use two angles, θ and φ, as shown in Fig. 5.3. Using a sphere radius 1, we can express the level of the $n - 1$ horizontal circles by their z-coordinate

$$z = \cos \varphi$$

Writing $\delta = \pi/n$, we will only use horizontal circles corresponding to the following angles:

$$\varphi = i \times \delta \qquad (i = 1, 2, \ldots, n - 1)$$

On each of these circles we have to use $2n$ vertices, which correspond to the angles

$$\theta = j \times \delta \qquad (j = 0, 1, \ldots, 2n - 1)$$

Thus, each pair (i, j) is associated with a vertex, so that we can devise a means of associating a vertex number with it. Since 1 has been used for the north pole, we start with vertex number 2 on circle $i = 1$. With $2n$ vertices on each horizontal circle, the first vertex number available for circle $i = 2$ will be $2n + 2$, and for circle $i = 3$ it will be $4n + 2$, and so on. In general, on circle i, we begin with number $(i - 1) \cdot 2n + 2$. Since on each circle there are $2n$ vertices, identified as $j = 0, \ldots, 2n - 1$, we have

$$\text{number for vertex } (i, j) = (i - 1) \times 2n + 2 + j$$

As we have seen in Sect. 5.2, the x-, y- and z-coordinates for this vertex (i, j) is computed as $x = \sin \varphi \cos \theta$, $y = \sin \varphi \sin \theta$ and $z = \cos \varphi$, where θ and φ depend upon i and j as shown above.

Finally, we have to assign a vertex number to the south pole. As we already have used $1 + (n - 1) \cdot 2n$ vertex numbers, the one for the south pole will be

$$1 + (n - 1) \times 2n + 1 = 2\left(n^2 - n + 1\right)$$

which is at the same time the total number of vertices.

5.8 Analyze this problem in the same way as was done for Exercise 5.7. Here each triangle (at the south pole) and each parallelogram is to be specified both counter-clockwise and clockwise, since either side of the curved surface can in principle be visible. Unlike Exercise 5.7, we had better use the variable n for the number of slices of *half* the sphere in this problem, so that there are $4n$ instead of $2n$ vertices on every horizontal circle that we use, giving altogether $n \cdot 4n + 1 = 4n^2 + 1$ vertices.

5.9 Use program arguments for the numbers of squares in each of the three directions x, y and z. Remember, the word *Faces* can occur only once in the file, so we have to specify the vertices of *all* cubes before we start specifying the faces.

5.10 Let us start with a torus such as the one in Sect. D.3, that is, a horizontal one with O as its center, and let the second torus be a vertical one, with its center on the positive x-axis. We will make the sizes of the tori and their numbers of vertices identical; only their positions are different. As in Sect. D.3, the size of a torus (and its shape) is completely determined by the radii R and r, where $r = 1$. Since the hole in each torus must be wide enough for the other to pass through, it is required that $R \geq 2r$, that is, $R \geq 2$.

For each vertex of the first torus, there is a corresponding one on the second. As we have seen in Sect. D.3 there are n^2 vertices for each torus, so that we can use the numbers i and $i + n^2$ for each pair of corresponding vertices. To obtain the second torus, we have to shift the first one a distance R towards the positive x-axis, after which we turn it through the x-axis though 90°. (Because of this special angle, no complicated computations are required for this rotation, so it is not worthwhile to use the class *Rota3D* of Sect. 3.9 in this case.) Writing (x, y, z) for vertex i and (x', y', z') for the corresponding vertex $i + n^2$, we now have $x' = x + R$, $y' = -z$, $z' = y$.

Although, in the first part of 3D data files, we usually supply the vertices in ascending order of the vertex numbers, this is not required. It is therefore possible to write pairs of lines of the following form in the first part of the data file:

$$\begin{array}{llll} i & x & y & z \\ i+n^2 & x' & y' & z' \end{array}$$

Similarly, in the second part of the file, we can write faces in pairs, with vertex numbers in the second face of each pair n^2 higher than those of the first of that pair. In this way, the desired program for two tori can be obtained from *Torus.java* by adding only a few statements.

5.11 If necessary, you might refer to the method *genCylinder* in program *Cylinder.java* of Sect. 5.8 for the cylindrical pole in the middle of the staircase. If you do, bear in mind that the situation here is simpler because this cylinder is solid (as is the case in program *Cylinder.java* with *rInner* = 0). You can use program *Beams.java* of Sect. D.4 to see how the steps can be constructed, or you can use the class *Rota3D* (see also Exercise 5.5), provided that you also perform a translation, adding a constant to the z-coordinates of each new step. As for the railing, recall that the data file, after the word *Faces*, can contain line segments specified as two vertex numbers followed by a period, as we have discussed in Sect. 5.5.

5.12 This part of the game needs to be runnable by generating shapes randomly and moving the current shape automatically in a constant speed. Rather than using a button, we could draw a rectangle and detect the position (i.e. the x and y coordinates) of each click to determine if the user has clicked the button. All the coordinates in the game should be relative to every component (rather than absolute).

Suppose the main class is *TetrisMoving* and we use the canvas class *CvTetrisMoving*. Then the following fragments show how to react to mouse wheel events in addition to left and right mouse clicks:

```
import java.awt.event.*;
...
class CvTetrisMoving extends Canvas
   implements Runnable, MouseListener, MouseWheelListener {
   ...
   CvTetrisMoving() { // Constructor
      addMouseListener(this);
      addMouseWheelListener(this);
      ...
   }
   public void mousePressed(MouseEvent e) {
      int x = e.getX(), y = e.getY();
      ...
      boolean
         leftButton =
            (e.getModifiers() & InputEvent.BUTTON1_MASK) != 0,
         rightButton =
            (e.getModifiers() & InputEvent.BUTTON3_MASK) != 0;
         ...
      }
   }
   public void mouseReleased(MouseEvent e) {}
   public void mouseEntered(MouseEvent e) {}
   public void mouseClicked(MouseEvent e) {}
   public void mouseExited(MouseEvent e) {}

    public void mouseWheelMoved(MouseWheelEvent e) {
       int rot = e.getWheelRotation();
       // scroll downward: rot = +1
       // scroll upward: rot = -1
       ...
    }
    ...
 }
```

As for animation, refer to the hint given for Exercise 5.4.

6.1 The desired input file is listed below:

```
1  1 -1 0
2  1  1 0
3 -1  1 0
```

```
4 -1 -1 0
5  0  0 -2
6  0  0 2
Faces:
1 2 3 4.
4 3 2 1.
5 6.
```

6.2 Use the vertices 1, 2 and 3 as triangle vertices in the plane $z = 0$, such that the origin O lies inside the triangle. Let vertex 4 be the origin and vertices 5 and 6 the same line endpoints as in the above solution to Exercise 6.1. Then, when specifying the triangle, use the invisible lines 1-4, 2-4 and 3-4 in the same way as the line 7-10 in Fig. 5.13. In other words, define each of the two sides of the triangle as a rather complex polygon, specified as a sequence of ten numbers by visiting, for example, the vertices 1, 2, 4, 2, 3, 4, and so on, in that order, using some minus signs in the same way as was done for Fig. 5.13.

6.3 See Sect. 5.5 for holes in polygons. Fig. 6.17 was obtained by using a data file of the same structure as the above one (see Exercise 6.1), but with 16 vertices and four faces. Based on Fig. E.2, the first of these faces was specified as follows:

```
1 2 3 -7 6 5 8 7 -3 4.
```

6.4 A simple solution to this problem is obtained by adding some code to draw *all* polygon edges (visible as well as invisible) as dashed lines. In addition to this, the visible edges are drawn as solid lines without any modification to the hidden-line algorithm. In other words, every visible edge is drawn as coinciding solid and dashed lines, which gives the effect of a solid line. Although HP-GL provides the command *LT* (Line Type) to draw dashed lines, we obtain better results if we draw our own, computed dashes, which are required for screen output anyway. Note that every dashed line in Fig. 6.18 begins and ends with a dash of the same length as the other ones. To implement all this, use the following file *HLinesDashed.java* instead of *HLines.java*:

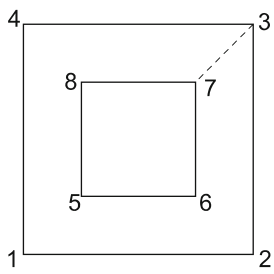


Fig. E.2 One of the four faces for the square rings of Fig. 6.17

```
// HLinesDashed.java: Perspective drawing with
// hidden-line elimination.
// Hidden lines are drawn as dashed lines.
import java.awt.*;

public class HLinesDashed extends Frame {
   public static void main(String[] args) {
      new Fr3DHDashed(args.length > 0 ? args[0] : null,
           new CvHLinesDashed(), "Hidden-lines dashed");
   }
}
```

Class *Fr3DHDashed* is defined in the following file:

```
// Fr3DHDashed.java: Frame class for HLinesDashed.java.
// This class extends Fr3D to enable writing HP-GL output files.
import java.awt.*;
import java.awt.event.*;

class Fr3DHDashed extends Fr3D {
   private MenuItem exportHPGL;
   CvHLinesDashed cv;

   Fr3DHDashed(String argFileName, CvHLinesDashed cv,
       String textTitle) {
       super(argFileName, cv, textTitle);
       exportHPGL = new MenuItem("Export HP-GL");
       mF.add(exportHPGL);
       exportHPGL.addActionListener(this);
       this.cv = cv;
   }

   public void actionPerformed(ActionEvent ae) {
       if (ae.getSource() instanceof MenuItem) {
          MenuItem mi = (MenuItem)ae.getSource();
          if (mi == exportHPGL) {
             Obj3D obj = cv.getObj();
             if (obj != null) {
                cv.setHPGL(new HPGL(obj));
                cv.repaint();
             }
             else
               Toolkit.getDefaultToolkit().beep();
          }
          else
```

```
                super.actionPerformed(ae);
            }
        }
    }
```

Use a copy of the file *CvHLines.java*, change the class name into *CvHLinesDashed* and apply further modifications as follows.

(a) Disable back-face culling by deleting the following program line in the method *buildLineSet*.

```
if (n > 2 && pol.getH() > 0) continue;
```

(b) Add the following method to the class:

```
void dashedLine(Graphics g, float xA, float yA,
                   float xB, float yB, float dashLength) {
  float u1 = xB - xA, u2 = yB - yA,
        len = (float)Math.sqrt(u1 * u1 + u2 * u2);
  int n = Math.round((len/dashLength + 1)/2);
  float h1 = u1/(2 * n - 1), h2 = u2/(2 * n - 1);
  for (int i=0; i<n; i++) {
     float x1 = xA + 2 * i * h1, y1 = yA + 2 * i * h2,
           x2 = x1 + h1, y2 = y1 + h2;
     drawLine(g, x1, y1, x2, y2);
     if (hpgl != null) {
         hpgl.write("PU;PA" + hpx(x1) + "," + hpy(y1) + ";");
        hpgl.write("PD;PA" + hpx(x2) + "," + hpy(y2) + ";\n");
     }
   }
}
```

(c) At the beginning of method *lineSegment*, insert the following if-statement:

```
if (iStart == 0)
dashedLine(g, pScr.x, pScr.y, qScr.x, qScr.y, 5);
```

6.5 In Fig. E.3, the two outer faces on the left and right are parallel, but the corresponding inner faces are not, as the distances 18 and 19 indicate. The latter faces, which are visible here, become invisible if we view the object from very far away, as Fig. E.4 shows.

Recall that, with eye-coordinates, the x-axis points to the right, the y-axis upwards and the z-axis towards us. Let us focus on the inner face that is visible

Fig. E.3 Object nearby: two inner faces visible

Fig. E.4 The same object far away: two inner faces invisible

in Fig. E.3 on the right but invisible in Fig. E.4. Estimating the normal vector $\mathbf{n} = (a, b, c)$ and the value h, as specified in the Exercise, for this face, we find:

a is almost equal to -1, because $\mathbf{n}$ almost points toward the negative x-axis;
b is almost zero, but positive because we view the object slightly from above;
c is almost zero, but negative because $\mathbf{n}$ points a little to the back.

These values, and in particular c, are independent of the viewing distance. By contrast, the inner product $h = \mathbf{n} \cdot \mathbf{x}$, where $\mathbf{x}$ is a vector from the viewpoint E to any point of the face in question, depends on the viewing distance. You should verify this by drawing a sketch similar to Fig. 6.19 but applied to this example. As a result, you will find that h is negative in Fig. E.3 but positive in Fig. E.4. This example demonstrates that, to determine if a face is a backface, we should use the sign of h, not that of c. The correct practice of using h for this purpose is equivalent to back-face culling based on the orientation of three points: if this orientation on the screen is the same as when the object is viewed from outside, the face is visible. Using c instead of h would be equivalent to using the eye-coordinates x and y instead of the screen coordinates in determining the orientation of image points. This would work correctly for most situations, but it may result in wrongly deciding that faces are invisible, especially if the object is viewed from nearby, as Fig. E.3 illustrates.

6.6 We can use back-face culling to decide which faces of the cube are visible. Refer to the solutions to Exercises 5.4 and 5.5 for the implementation of animation and rotation, respectively.

6.7 In the previous exercise we could have realized the effect of a rotation about a vertical axis by changing the angle θ and leaving the cube unchanged. This is no longer the case here because we now want to use two rotations. Fig. 6.20 was obtained by rotating each cube about one of its vertical edges, with different rotation speeds, the latter simply meaning that we use different angles in each step. As in the solution to Exercise 5.5, you need to supply only one method *run*, in which only one infinite loop occurs.

6.8 Change the class *CvPaint* (in the program file *Painter.java*) as follows:

(a) At the beginning of the class *CvPaint*, before {, add the line

```
implements Runnable
```

(b) After {, add the following lines:

```
Image image;
Graphics gImage;
double sunTheta = 0;
Thread thr = new Thread(this);
```

(c) Insert the method *run*, similar to the one given above for Exercise 5.4, but containing statements to update the spherical coordinate *sunTheta* and the variables *obj.sunX* and *obj.sunY*; you can use a constant value for *sunPhi*, which makes *obj.sunZ* also a constant. By using spherical coordinates, with radius $\rho = 1$, the light vector will have length 1.

(d) Insert the program lines

```
int w, h;
CvPainter(){thr.start();}
public void update(Graphics g){paint(g);}
```

(e) Modify the *paint* method, using the variables *image* and *gImage* besides *g*, in about the same way as was done in program *Anim.java*, listed above as help for Exercise 5.4.

6.9 For the format of the desired file see the file for Exercise 6.1. As for the program to generate 3D data files, refer to Sect. 5.8 and Appendix D, if necessary.

7.1 After defining

```
final int WHITE = 0xFFFFFF, MAGENTA = 0xFF00FF,
YELLOW = 0xFFFF00, CYAN = 0x00FFFF;
```

it will be clear that the color of the upper part of the upper circle can be set by

```
g2.setColor(new Color(MAGENTA));
```

For the red intersection (see Fig. 7.2b), we can demonstrate the subtractive nature of the color system by obtaining the color red as the difference of *WHITE* and *MAGENTA* ^ *YELLOW*. Here ^ is the bitwise exclusive-OR operator: 0 ^ 0 = 0, 0 ^ 1 = 1, 1 ^ 0 = 1, 1 ^ 1 = 0.

Using this, we can write

```
int red = WHITE - (MAGENTA ^ YELLOW);
g2.setColor(new Color(red));
g2.fill(intersectTopLeft);
```

The colors green and blue for the other two intersections can be computed from *WHITE*, *MAGENTA*, *YELLOW*, and *CYAN* in a similar way.

7.2 With screen coordinates of a triangle ABC stored in the arrays *x* and *y*, and for some not too small *int* value *n*, you may generate a great many points P inside the triangle by interpolation among A, B and C as follows:

```
for (int ia=0; ia<=n; ia++)
  for (int ib = 0; ib<=n-ia; ib++){
     int ic = n - ia - ib;
     double a = ia/(double)n, b = ib/(double)n, c = ic/(double)n;
     // a + b + c = 1, a, b, and c non-negative
     int xP = (int) (a * x[0] + b * x[1] + c * x[2] + 0.5),
         yP = (int) (a * y[0] + b * y[1] + c * y[2] + 0.5);
     ...
  }
```

Since the colors of the vertices A, B and C are given (see Fig. 7.5), you can compute the R, G, and B values of point P by interpolating in a similar way. Then fill a tiny rectangle at P with this computed color. You may do this in a modified version of class *CvPainter*, see Sect. 6.3.

7.3 You may adapt and combine the programs *Transparency.java* and *Texture.java* in Sects. 7.5 and 7.6, while properly position the three letters A so that they overlap to show the transparency effects.

7.4 Try several values for the constants used in the method *setSpecular*, to obtain pleasing results.

8.1 In the program *FractalGrammars.java*, there is the following fragment, which draws a line from the current point (*xLast*, *yLast*) to the new point (*xLast* + *dx*, *yLast* + *dy*), which, after the call to *drawTo*, will automatically be the current point (*xLast*, *yLast*).

```
case 'F': // Step forward and draw
   // Start: (xLast, yLast), direction: dir, steplength: len
   if (depth == 0) {
       double rad = Math.PI/180 * dir, // Degrees -> radians
       dx = len * Math.cos(rad), dy = len * Math.sin(rad);
       drawTo(g, xLast + dx, yLast + dy);
   }
```

Besides *xLast* and *yLast*, introduce the variables *xCorner* and *yCorner*, indicating the cornerpoints that we will not really visit because of the rounded corners. Each time, instead of drawing a line as discussed above, draw two lines. The first is one from the current point (*xLast*, *yLast*) to (*xCorner* + *dx*/4, *yCorner* + *dy*/4) to approximate the rounded corner. After this, the point just mentioned is now automatically stored as the new point (*xLast*, *yLast*) to enable you to used *drawTo* again. Then update the variables *xCorner* and *yCorner* by increasing them by *dx* and *dy*, respectively, so they indicate the next cornerpoint. Then you draw the second line, from the current point (*xLast*, *yLast*) to (*xCorner* − *dx*/4, *yCorner* − *dy*/4). Note that this last line is half as long as the full line drawn in the above fragment, since a quarter of it at the beginning and a quarter of it at the end are now replaced with the approximated rounded corners.

8.2 In the *paint* method of the program *Koch.java* in Sect. 8.2, there is only one call to *drawKoch* preceded by setting $dir = 0$. All you have to do is adding two other such calls, each preceded by assigning an appropriate value to *dir* so that the turtle starts in the right direction.

8.3 This hint is based on the program *FractalGrammars.java* and the string grammar *Tree2* of Sect. 8.3. In this example, we have

$$strX = {}^{''}F[+X]F[-X]+X^{''}$$

Each branch was drawn as a filled polygon instead of as a simple line segment by the following modified *drawTo* method:

```
void drawTo(Graphics g, double x, double y) {
   double r = rLast * 0.9;
   double dx = x - xLast, dy = y - yLast;
   double h = rLast * dy, v = rLast * dx, h1 = r * dy, v1 = r * dx;
   double [] xPol = {xLast + h, x + h1, x - h1, xLast - h},
            yPol = {yLast - v, y - v1, y + v1, yLast + v};
   int xDev[] = new int[4], yDev[] = new int[4];
```

```
      for (int i=0; i<4; ++i) {
         xDev[i] = iX(xPol[i]);
         yDev[i] = iY(yPol[i]);
      }
      g.fillPolygon(xDev, yDev, 4);
      xLast = x;
      yLast = y;
      rLast = r;
   }
```

8.4 Using the same string *strX* as in Exercise 8.3, each time the second *F* in this string is encountered a branch is drawn that should have a leaf at its end. So in the switch statement you should add a fragment to draw a leaf in the *case F* part after the call to *drawTo*, provided that the position counter *i* for *strX* is equal to 5. One way of drawing a closed figure that approximates the shape of a leaf is by drawing a sequence of filled circles (by means of *drawOval*) whose centers lie on a line that has the same direction (*dx*, *dy*) as the branch in question.

8.5 Use methods *iX* and *iY* to convert logical to device coordinates and methods *fx* and *fy* for the inverse conversions. Restricting this discussion to *x*-coordinates, we can use

```
int iX(float x) {
  return (int)(xDevCenter + (x - xLogCenter)/pixelSize);
}

float fx(int x) {
  return xLogCenter + (x - xDevCenter) * pixelSize;
}
```

As usual, we use *d* defined as

```
Dimension d = getSize();
```

Let us denote the current boundaries of the logical *x*-coordinates by *xLeft* and *xRight*.

For example, we can initially set these boundaries equal to those of the device coordinates, that is, to 0 and *d.width*(), respectively. In the method *mouseReleased*, we obtain the device coordinates *xs* and *xe* for the left and right boundaries of the cropping rectangle. We then apply the method *fx* to these to obtain the corresponding logical coordinates, writing, for example,

```
xLeftNew = fx(xs);
xRightNew = fx(xe);
```

Then these new values are assigned to *xLeft* and *xRight*, and then used to compute

```
pixelSize = Math.max((xRight - xLeft)/d.width,
                     (yTop - yBottom)/d.height);
xLogCenter = (xLeft + xRight)/2;
```

Let us now discuss the plausibility of these statements (rather than proving them rigorously). Normally, *mouseRelease* provides us with a range (*xs*, *xe*) that is smaller than the width of the drawing rectangle. Then after applying *fx* and *fy*, the new logical *x*-range (*xLeft*, *xRight*) will also decrease, and the same applies to *pixelSize*. As a result of the latter, the value added to *xDevCenter* in the above method *iX* will be larger than it was before, so that the figure will appear on a larger scale. As for panning, let us assume that the new x-range selected by the user is on the left half of the screen. Then the new center *xLogCenter* of the logical *x*-range will be smaller than it was before, which will increase the value computed by the method *iX*. This should indeed be the case, since the part of the image displayed in the selected *x*-range on the left half of the screen should be displayed in the center of the drawing rectangle, or, in other words, it should shift to the right.

8.6 Modify the *MandelbrotZoom.java* program and the *paint* method for Julia sets. Combine the two programs so that the latter will draw Julia sets in a side window.

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